Algorithms and Data Structures

A13. Sorting: Lower Bound

Gabriele Röger and Patrick Schnider

University of Basel

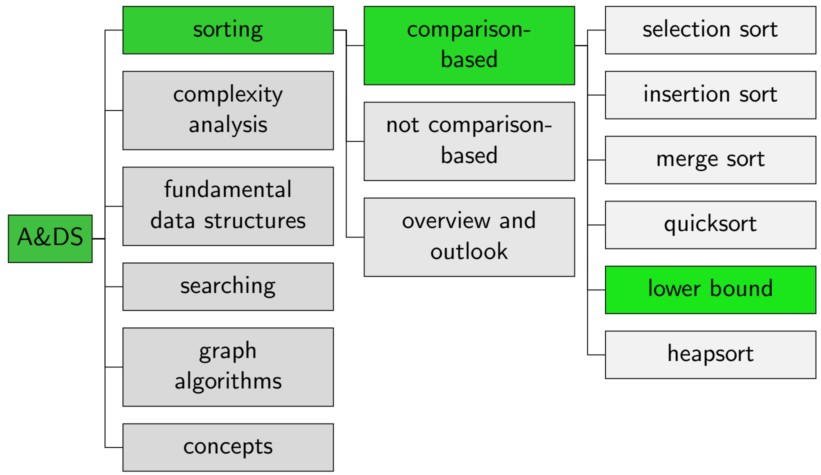
March 19, 2025

A13.1 Lower Bound on Necessary Comparison Operations

A13.2 Summary

A13.1 Lower Bound on Necessary Comparison Operations

Content of the Course



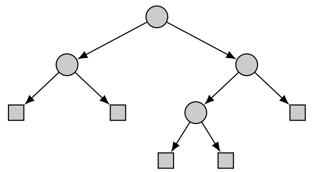
Question

- So far, merge sort and heapsort had with $O(n \log_2 n)$ the best (worst-case) running time.
- Can we do better?
- We show: Not with comparison-based approaches!

How we Proceed

- Difficulty: We cannot analyze a specific algorithm but must make an argument for all possible approaches.
- Comparison-based approaches can only analyze the input by means of key comparisons.
- They must sort every input correctly.
- From this, we can derive a lower bound on the number of key comparisons in the worst case.

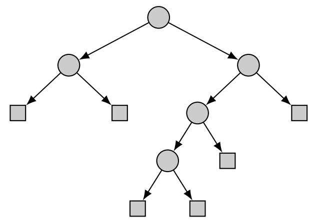
Crash Course: Binary Trees



- Binary tree: each node has at most two successor nodes.
- Nodes without successors are called leaves (squares in image).
- The node without a predecessor (at the top) is the root.
- The depth of a leaf is the number of edges from the root to the leaf.

The maximal depth of a leaf in a binary tree with k leaves is at least $\log_2 k$.

Exercise (Slido)

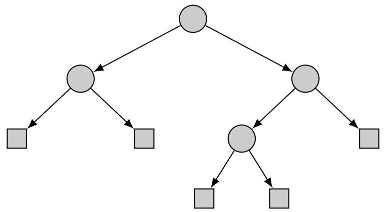


What is the maximal depth of a leaf in this tree?

Abstract Behavior as Tree

Consider an arbitrary comparison-based sorting algorithm A.

- Its behavior only depends on the results of key comparisons.
- For each key comparison, there are two possibilities how the algorithm proceeds.
- For an input of a given size, we can depict this graphically as a decision tree.



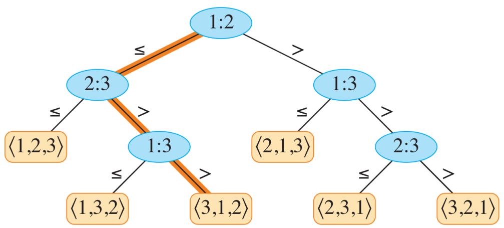
- Execution of A corresponds to tracing a simple path from the root down to a leaf.

Result as Permutation

What does the algorithm have to be able to do?

- Assumption: all input elements distinct.
- Must sort all input sequences of size n correctly.
- We can adapt all algorithms so that they trace from which position to which position they move the elements.
- Then the result is not the sorted array, but the corresponding permutation.
- Since all possible inputs of size n must be sorted correctly, the algorithm must be able to generate all $n!$ possible permutations.

Example: Tree for Insertion Sort on Three Elements



Highlighted path e.g. for sorting sequence $[a_1 = 6, a_2 = 8, a_3 = 5]$

Source: Cormen et al., Introduction to Algorithms

Lower Bound I

- Each leaf in the tree corresponds to one permutation.
- For input size n, the tree must thus have at least $n!$ leaves.
- The maximal depth of a leaf in the tree is therefore $\geq \log_2(n!)$.
- There is an input of size n with $\geq \log_2(n!)$ key comparisons.

Lower Bound II

Lower bound on $\log_2(n!)$

- It holds that $n! \geq (\frac{n}{2})^{\frac{n}{2}}$
  $4! = 1 \cdot 2 \cdot \underbrace{3}_{\geq 2} \cdot \underbrace{4}_{\geq 2} \geq 2^2$
- $\log_2(n!) \geq log_2((\frac{n}{2})^{\frac{n}{2}}) = \frac{n}{2} log_2(\frac{n}{2})$
  $= \frac{n}{2}(\log_2 n + \log_2 \frac{1}{2}) = \frac{n}{2}(\log_2 n - \log_2 2)$
  $= \frac{n}{2}(\log_2 n - 1)$

Theorem

Every comparison-based sorting algorithm requires $\Omega(n \log n)$ key comparisons in the worst case. As a result, also the worst-case running time is $\Omega(n \log n)$.

Heapsort and merge sort are asymptotically optimal.

A13.2 Summary

Summary

- Every comparison-based sorting algorithm has at least linearithmic worst-case running time.