Algorithms and Data Structures

A10. Runtime Analysis: Divide-and-Conquer Algorithms

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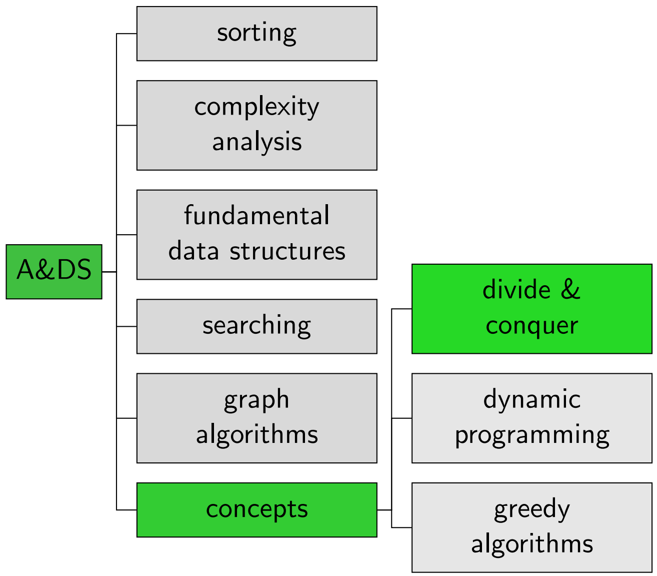
A10.1 Divide-and-Conquer Algorithms

Content of the Course



Recap: Merge Sort

Sort input range with n elements:

- $n \leq 1$: nothing to do
- $n > 1$: proceed as follows:

Divide the range into two roughly equally-sized ranges.
Conquer each of them by recursively sorting them.
Combine the sorted subranges to a fully sorted range.

Divide-and-Conquer Algorithm Scheme

Base case: If the problem is small enough, solve it directly without recursing.

Recursive case: Otherwise

Divide the problem into one or more subproblems that are smaller instances of the same problem.
Conquer the subproblems by solving them recursively.
Combine the subproblem solutions to form a solution to the original problem.

Example: Multiplication of Square Matrices

Square matrix $A_{n \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{m2} & \cdots & a_{nn} \end{bmatrix}$

Let A, B be $n \times n$ matrices. We want to compute $C = A \cdot B$.

For $i, j \in \{1, \dots, n\}$: Set $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$.

Example: Multiplication of Square Matrices

Direct Computation

```python
def matrix_multiply(A, B, n):
    for i in range(1,n+1): # i = 1,...,n
        for j in range(1,n+1): # j = 1,...,n
            for k in range(1,n+1): # k = 1,...,n
                C[i][j] += A[i][k] * B[k][j]
```

Running time $\Theta(n^3)$

Example: Multiplication of Square Matrices
A Simple Divide-and-Conquer Algorithm

Assumption: $n = 2^k$ for some $k \in \mathbb{N}$.

Idea: Divide each matrix into four $n/2 \times n/2$ matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Can compute $C = A \cdot B$ as

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} & A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{bmatrix}$$

Eight $n/2 \times n/2$ multiplications and four $n/2 \times n/2$ additions

Example: Multiplication of Square Matrices
A Simple Divide-and-Conquer Algorithm

function MATRIX-MULTIPLY-RECURSIVE(A, B, n)
  if $n == 1$ **then**
    $c_{11} = a_{11} \cdot b_{11}$
    return
  partition A and B into $n/2 \times n/2$ submatrices
    $A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, \ldots, B_{22}$
    (details omitted; takes constant time)
  P_{1111} =MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11}, $n/2$)
    ...   (8 recursive calls total)
  P_{2222} =MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22}, $n/2$)
  $C_{11} = P_{1111} + P_{1221}$
    ...   (4 additions total)
  $C_{22} = P_{2112} + P_{2222}$

Example: Multiplication of Square Matrices
Strassen's Algorithm

- The previous algorithm still has running time $\Theta(n^3)$.
- Strassen's algorithm is similar but uses only 7 recursive calls.
- Idea (with scalars): Compute $x^2 + y^2$ as $(x + y)(x - y)$ with 2 additions, 1 multiplication instead of 2 multiplications, 1 addition
- Computes the four submatrices $C_{11}, C_{12}, C_{21}, C_{22}$ with four steps (next slide).

Example: Multiplication of Square Matrices
Strassen's Algorithm

Setting

$$P_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$$
$$P_2 = (A_{21} + A_{22}) \cdot B_{11}$$
$$P_3 = A_{11} \cdot (B_{12} - B_{22})$$
$$P_4 = A_{22} \cdot (B_{21} - B_{11})$$
$$P_5 = (A_{21} + A_{12}) \cdot B_{22}$$
$$P_6 = (A_{21} - A_{11}) \cdot (B_{11} + B_{12})$$
$$P_7 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$$

we can compute $C = A \cdot B$ as

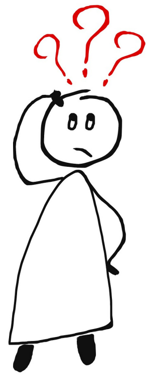
$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} P_1 + P_4 - P_5 + P_7 & P_3 + P_5 \\ P_2 + P_4 & P_1 - P_2 + P_3 + P_6 \end{bmatrix}$$

Example: Multiplication of Square Matrices
Strassen's Algorithm (Sketch)

1. If n is 1, proceeds as in MATRIX-MULTIPLY-RECURSIVE, otherwise, partition matrices A, B as in MATRIX-MULTIPLY-RECURSIVE. This takes $\Theta(1)$ time.
2. Create $n/2 \times n/2$ matrices $S_1, S_2, \dots, S_{10}$, each of which is the sum or difference of two submatrices from step 1. Create and zero the entries of seven $n/2 \times n/2$ matrices $P_1, \dots, P_7$ to hold seven matrix products (next step).
   All 17 matrices can be created/initialized in $\Theta(n^2)$ time.
3. Recursively compute each of the seven products $P_1, \dots, P_7$.
4. Update the four submatrices $C_{11}, \dots, C_{22}$ by adding or subtracting various P_i matrices. This takes $\Theta(n^2)$ time.

Running time $\Theta(n^{\lg 7})$ (with $\lg 7 \approx 2.8073549 < 3$)

Questions

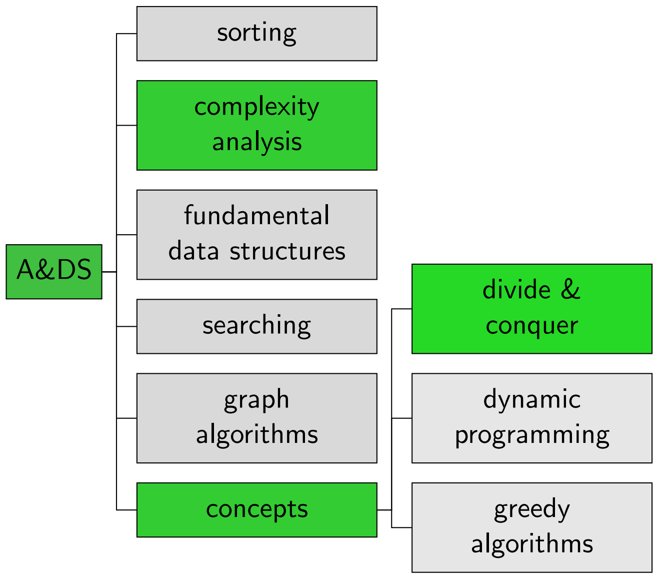


Your Questions?

How can we analyze the running time of such algorithms?

A10.2 Recurrences

Content of the Course



Recurrences

A recurrence is a recursively defined function $f : \mathbb{N}_0 \to \mathbb{R}$ where for almost all n, the value $f(n)$ is defined in terms of the values $f(m)$ for $m < n$.

Example (Fibonacci Series)

$$
\begin{aligned}
F(0) &= 0 && \text{(1st base case)}\\
F(1) &= 1 && \text{(2nd base case)}\\
F(n) &= F(n-2) + F(n-1) \text{ for all } n \geq 2 && \text{(recursive case)}
\end{aligned}
$$

Recurrences occur naturally for the running time of divide-and-conquer algorithms.

Example: Top-Down Merge Sort

```
1  def sort(array):
2      tmp = [0] * len(array)  # [0,...,0] with same size as array
3      sort_aux(array, tmp, 0, len(array) - 1)
4  
5  def sort_aux(array, tmp, lo, hi):
6      if hi <= lo:
7          return
8      mid = lo + (hi - lo) // 2
9      sort_aux(array, tmp, lo, mid)
10     sort_aux(array, tmp, mid + 1, hi)
11     merge(array, tmp, lo, mid, hi)
```

Analysis for $m = \text{hi} - \text{lo} + 1$

c_0 for lines 6–7

c_1 for lines 6–8

$c_2 m$ for merge step (takes linear time)

Example: Top-Down Merge Sort

Assumption: $n = 2^k$ for some $k \in \mathbb{N}$

Running time `sort_aux`

- $T(1) = c_0$
- $T(m) = c_1 + 2T(m/2) + c_2 m$

Example: Multiplication of Square Matrices

An Adapted Divide-and-Conquer Algorithm

The following algorithm computes $C = C + A \cdot B$:

function MATRIX-MULTIPLY-RECURSIVE(A, B, C, n)
    if $n == 1$ **then**
        $c_{11} = c_{11} + a_{11} \cdot b_{11}$
        return
    partition A, B, and C into $n/2 \times n/2$ submatrices
        $A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, \ldots, B_{22}, C_{11}, \ldots, C_{22}$
        (details omitted; takes constant time)
    MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11}, C_{11}, $n/2$)
        ... (8 recursive calls total)
    MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22}, C_{22}, $n/2$)

Example: Multiplication of Square Matrices
A Simple Divide-and-Conquer Algorithm

Assumptions:

- $n = 2^k$ for some $k \in \mathbb{N}$,
- c_0 is the running time in case $n = 1$, and
- c_1 is the time for the partition into submatrices.

Specify a recurrence for the running time $T(n)$ of the algorithm.

Solution:

$$T(1) = c_0$$
$$T(n) = c_1 + 8T(n/2) \quad \text{for } n > 1$$

Algorithmic Recurrences

A recurrence $T(n)$ is algorithmic if, for every sufficiently large $n_0 > 0$, the following two properties hold:

1. For all $n < n_0$, we have $T(n) = \Theta(1)$.
2. For all $n \geq n_0$, every path of recursion terminates in a defined base case within a finite number of recursive invocations.

Convention

- Whenever a recurrence is stated without an explicit base case, we assume that the recurrence is algorithmic.
- For non-recursive aspects, we use $\Theta(\cdot)$ (or $O(\cdot)$ if only interested in upper bound).

Examples:

- $T(m) = 2T(m/2) + \Theta(m)$
  for merge sort.
- $T(n) = 8T(n/2) + \Theta(1)$
  for simple recursive matrix multiplication.

Example: Multiplication of Square Matrices
Strassen's Algorithm (Sketch)

1. If n is 1, proceeds as in MATRIX-MULTIPLY-RECURSIVE, otherwise, partition matrices A, B, C as in MATRIX-MULTIPLY-RECURSIVE. This takes $\Theta(1)$ time.
2. Create $n/2 \times n/2$ matrices $S_1, S_2, \ldots, S_{10}$, each of which is the sum or difference of two submatrices from step 1. Create and zero the entries of seven $n/2 \times n/2$ matrices $P_1, \ldots, P_7$ to hold seven matrix products (next step).
   All 17 matrices can be created/initialized in $\Theta(n^2)$ time.
3. Recursively compute each of the seven products $P_1, \ldots, P_7$.
4. Update the four submatrices $C_{11}, \ldots, C_{22}$ by adding or subtracting various P_i matrices. This takes $\Theta(n^2)$ time.

$$T(n) = \Theta(1) + \Theta(n^2) + 7T(n/2) + \Theta(n^2) = 7T(n/2) + \Theta(n^2)$$

Summary

- Divide-and-conquer algorithms divide the problem into smaller problems of the same kind, solve them (typically recursively) and combine their solution into a solution of the full problem.
- Their running time can often easily be described with a recurrence.