Algorithms and Data Structures

A8. Runtime Analysis: Asymptotic Notation

Gabriele Röger and Patrick Schnider

University of Basel

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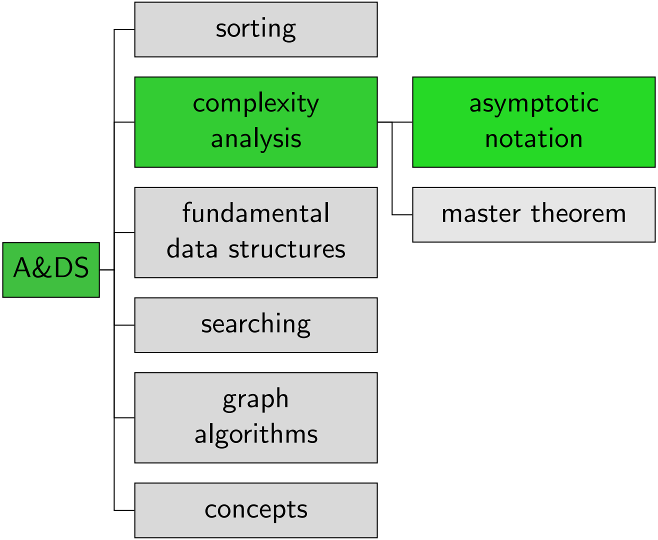
Asymptotic Notation

Content of the Course



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"The running time of merge sort grows asymptotically as fast as $n \log_2 n$."

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Bottom-up merge sort has linearithmic running time, i.e. there are constants $c, c', n_0 > 0$, such that for all $n \geq n_0$: $cn \log_2 n \leq T(n) \leq c' n \log_2 n$.

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- When determining the bounds, we ignored lower-order terms (constant and n) or let them disappear.
- We were not interested in the exact values of the constants but were satisfied if there exist some suitable constants.
- The running time for small n is not that important.

Previous Results

Theorem

The merge step has linear running time, i.e., there are constants $c, c', n_0 > 0$ such that for all $n \geq n_0$: $cn \leq T(n) \leq c'n$.

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Selection sort has quadratic running time, i.e., there are constants $c > 0, c' > 0, n_0 > 0$ such that for $n \geq n_0$: $cn^2 \leq T(n) \leq c'n^2$.

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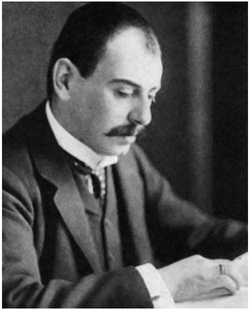
Can't we write this more compactly?

Asymptotic Notation/Landau-Bachmann Notation



Edmund Landau

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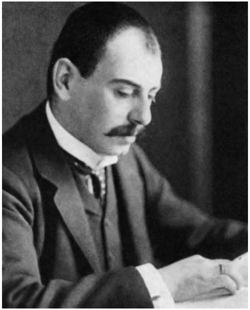

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Neutral term: Asymptotic notation
German: Landau notation
Internationally: Bachmann–Landau notation also after Paul Gustav Heinrich Bachmann (German mathematician)

Symbol Theta

Definition

For a function $g : \mathbb{N} \to \mathbb{R}$, we denote by $\Theta(g)$ the set of all functions $f : \mathbb{N} \to \mathbb{R}$ that grow asymptotically as fast as g:

$$\Theta(g) = \{f \mid \exists c > 0\ \exists c' > 0\ \exists n_0 > 0\ \forall n \geq n_0 : c \cdot g(n) \leq f(n) \leq c' \cdot g(n)\}$$

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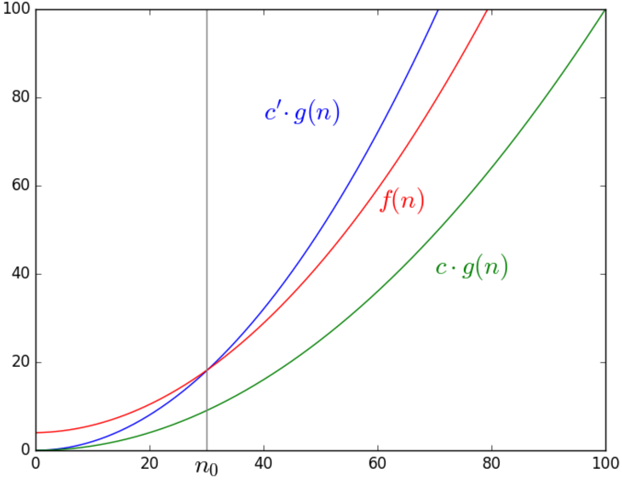
"The running time of merge sort is in $\Theta(n \log_2 n)$."
"$f \in \Theta(n^2)$ with $f(n) = 3n^2 + 5n + 39$"

or by convention (abusing notation/terminology) also

"The running time of merge sort is $\Theta(n \log_2 n)$."
"$3n^2 + 5n + 39 = \Theta(n^2)$"

Symbol Theta: Illustration

$$f \in \Theta(g)$$



Jupyter Notebook (with Exercises)

Jupyter notebook: `asymptotic_notation.ipynb`

More Symbols for Asymptotic Growth

- "f grows no faster than g."

$$O(g) = \{f \mid \exists c > 0\ \exists n_0 > 0\ \forall n \geq n_0 : f(n) \leq c \cdot g(n)\}$$

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Pronunciation: Θ: Theta, Ω: Omega, O: Oh

Less Frequently needed Symbols

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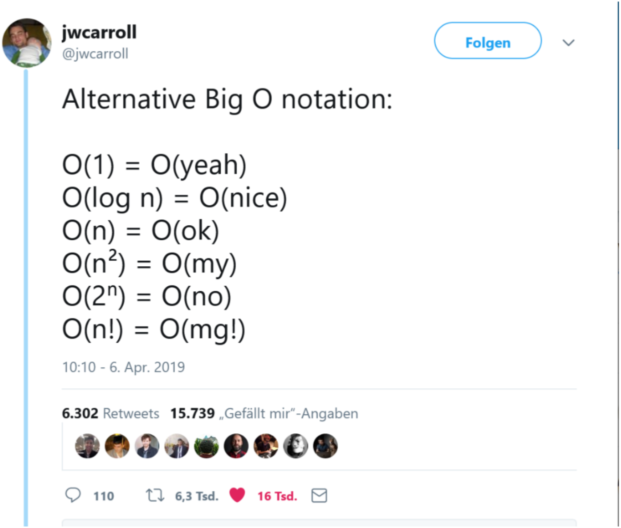
Pronunciation: ω: little-omega

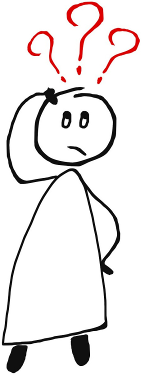
Some Relevant Classes of Functions

In increasing order (except for the general n^k):

g	growth
1	constant
$\log n$	logarithmic
n	linear
$n \log n$	linearithmic
n^2	quadratic
n^3	cubic
n^k	polynomial (constant k)
2^n	exponential


jwcarroll
@jwcarroll
Folgen
Alternative Big O notation:
O(1) = O(yeah)
O(log n) = O(nice)
O(n) = O(ok)
O(n²) = O(my)
O(2ⁿ) = O(no)
O(n!) = O(mg!)
10:10 - 6. Apr. 2019
6.302 Retweets 15.739 „Gefällt mir"-Angaben
110
6,3 Tsd.
16 Tsd.

Questions



Questions?

Rules

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  - $f_3(n) = 9n \log_2 n + n + 17$
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- Why is this the case?

Connections

It holds that:

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- $O(n^{k_1}) \subset O(n^{k_2})$ for $k_1 < k_2$
  e.g. $O(n^2) \subset O(n^3)$

Calculation Rules

- Product
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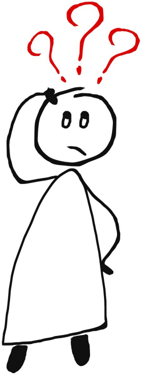
- Sum
  $f_1 \in O(g_1)$ and $f_2 \in O(g_2) \Rightarrow f_1 + f_2 \in O(g_1 + g_2)$
- Multiplication with a constant
  $k > 0$ and $f \in O(g) \Rightarrow kf \in O(g)$
  $k > 0 \Rightarrow O(kg) = O(g)$

Reason for Sufficiency of Highest-order Term

Example: $5n^3 + 2n \in O(n^3)$

- Due to rule for multiplication with a constant:
  - $5n^3 \in O(n^3)$
  - $2n \in O(n)$
- Because of $2n \in O(n)$ and $O(n) \subset O(n^3)$:
  - $2n \in O(n^3)$
- Sum rule:
  - $5n^3 + 2n \in O(n^3 + n^3)$
- Multiplication with a constant (for a class):
  - $5n^3 + 2n \in O(n^3)$

Questions



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Summary

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- With asymptotic notation, we refer to classes of functions that grow no faster/no slower/... than a function g.
  - $O(g)$: Growth no faster than g.
  - $\Theta(g)$: Growth asymptotically as fast as g.