

Algorithms and Data Structures

A8. Runtime Analysis: Asymptotic Notation

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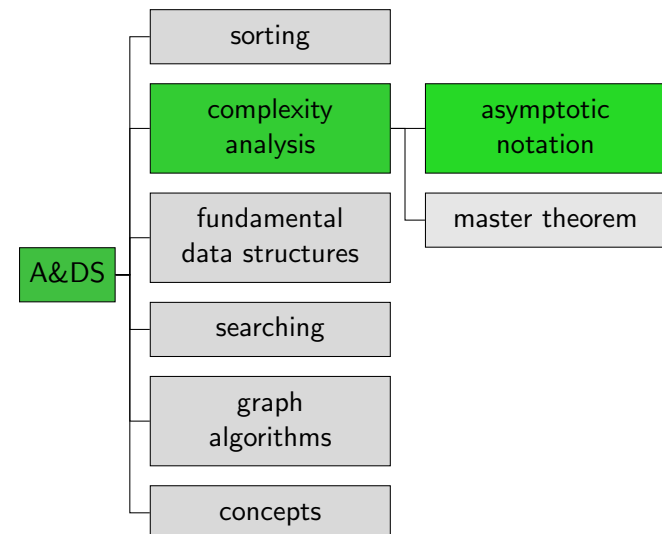
A8.1 Asymptotic Notation

A8.2 Rules

A8.3 Summary

A8.1 Asymptotic Notation

Content of the Course



Result for Merge Sort

“The running time of merge sort grows asymptotically as fast as $n \log_2 n$.”

Theorem

Bottom-up merge sort has *linearithmic running time*, i.e. there are constants $c, c', n_0 > 0$, such that for all $n \geq n_0$: $cn \log_2 n \leq T(n) \leq c'n \log_2 n$.

- ▶ When determining the bounds, we ignored lower-order terms (constant and n) or let them disappear.
- ▶ We were not interested in the exact values of the constants but were satisfied if there exist some suitable constants.
- ▶ The running time for small n is not that important.

Previous Results

Theorem

The merge step has *linear running time*, i.e., there are constants $c, c', n_0 > 0$ such that for all $n \geq n_0$: $cn \leq T(n) \leq c'n$.

Theorem

Merge sort has *linearithmic running time*, i.e. there are constants $c, c', n_0 > 0$, such that for all $n \geq n_0$: $cn \log_2 n \leq T(n) \leq c'n \log_2 n$.

Theorem

Selection sort has *quadratic running time*, i.e., there are constants $c > 0, c' > 0, n_0 > 0$ such that for $n \geq n_0$: $cn^2 \leq T(n) \leq c'n^2$.

Can't we write this more compactly?

Asymptotic Notation/Landau-Bachmann Notation



Edmund Landau

- ▶ German mathematician (1877–1938)
- ▶ analytic number theory
- ▶ no friend of applied mathematics

Neutral term: *Asymptotic notation*

German: *Landau notation*

Internationally: *Bachmann–Landau notation* also after Paul Gustav Heinrich Bachmann (German mathematician)

Symbol Theta

Definition

For a function $g : \mathbb{N} \rightarrow \mathbb{R}$, we denote by $\Theta(g)$ the set of all functions $f : \mathbb{N} \rightarrow \mathbb{R}$ that *grow asymptotically as fast as g* :

$$\Theta(g) = \{f \mid \exists c > 0 \exists c' > 0 \exists n_0 > 0 \forall n \geq n_0 : c \cdot g(n) \leq f(n) \leq c' \cdot g(n)\}$$

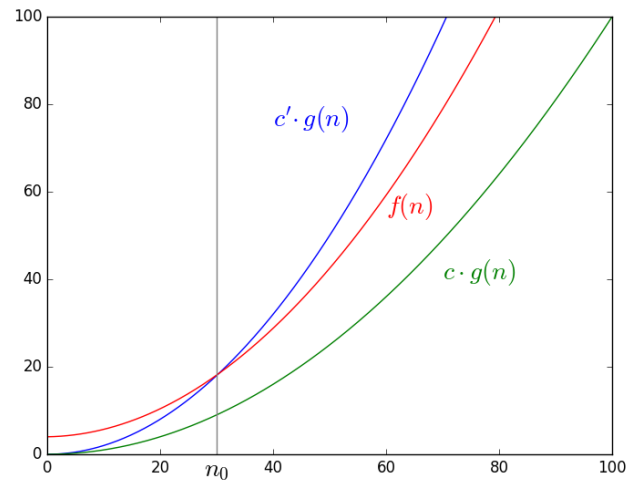
“The running time of merge sort is in $\Theta(n \log_2 n)$.”
“ $f \in \Theta(n^2)$ with $f(n) = 3n^2 + 5n + 39$ ”

or by convention (abusing notation/terminology) also

“The running time of merge sort is $\Theta(n \log_2 n)$.”
“ $3n^2 + 5n + 39 = \Theta(n^2)$ ”

Symbol Theta: Illustration

$$f \in \Theta(g)$$



Jupyter Notebook (with Exercises)



Jupyter notebook: `asymptotic_notation.ipynb`

More Symbols for Asymptotic Growth

- ▶ “ f grows no faster than g .”

$$O(g) = \{f \mid \exists c > 0 \exists n_0 > 0 \forall n \geq n_0 : f(n) \leq c \cdot g(n)\}$$

- ▶ O for “Ordnung” (order) of the function.
- ▶ “ f grows at least as fast as g .”

$$\Omega(g) = \{f \mid \exists c > 0 \exists n_0 > 0 \forall n \geq n_0 : c \cdot g(n) \leq f(n)\}$$

- ▶ $\Theta(g) = O(g) \cap \Omega(g)$
- ▶ $f \in \Omega(g)$ if and only if $g \in O(f)$.
- ▶ In computer science, we are often only interested in an upper bound on the growth of the running time: O instead of Θ

Pronunciation: Θ : Theta, Ω : Omega, O : Oh

Less Frequently needed Symbols

- ▶ “ f grows slower than g .”

$$o(g) = \{f \mid \forall c > 0 \exists n_0 > 0 \forall n \geq n_0 : f(n) \leq c \cdot g(n)\}$$

- ▶ “ f grows faster than g .”

$$\omega(g) = \{f \mid \forall c > 0 \exists n_0 > 0 \forall n \geq n_0 : c \cdot g(n) \leq f(n)\}$$

Pronunciation: ω : little-omega

Some Relevant Classes of Functions

In increasing order (except for the general n^k):

g	growth
1	constant
$\log n$	logarithmic
n	linear
$n \log n$	linearithmic
n^2	quadratic
n^3	cubic
n^k	polynomial (constant k)
2^n	exponential



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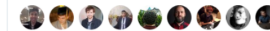
Folgen

Alternative Big O notation:

$O(1) = O(\text{yeah})$
 $O(\log n) = O(\text{nice})$
 $O(n) = O(\text{ok})$
 $O(n^2) = O(\text{my})$
 $O(2^n) = O(\text{no})$
 $O(n!) = O(\text{mg!})$

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A8.2 Rules

Examples for Θ

- ▶ In the analysis, only the highest-order term (= fastest-growing summand) of a function is relevant.
- ▶ Examples:
 - ▶ $f_1(n) = 5n^2 + 3n - 9 \in \Theta(n^2)$
 - ▶ $f_2(n) = 3n \log_2 n + 2n^2 \in \Theta(n^2)$
 - ▶ $f_3(n) = 9n \log_2 n + n + 17 \in \Theta(n \log n)$
 - ▶ $f_4(n) = 8 \in \Theta(1)$

Examples for Big-O

- ▶ In the analysis, only the highest-order term (= fastest-growing summand) of a function is relevant.
- ▶ Examples:
 - ▶ $f_1(n) = 8n^2 - 3n - 9 \in O(n^2)$
 - ▶ $f_2(n) = n^3 - 3n \log_2 n \in O(n^3)$
 - ▶ $f_3(n) = 3n \log_2 n + 1000n + 10^{200} \in O(n \log n)$
- ▶ Why is this the case?

Connections

It holds that:

- ▶ $O(1) \subset O(\log n) \subset O(n) \subset O(n \log n) \subset O(n^k) \subset O(2^n)$
(for $k \geq 2$)
- ▶ $O(n^{k_1}) \subset O(n^{k_2})$ for $k_1 < k_2$
e.g. $O(n^2) \subset O(n^3)$

Calculation Rules

- ▶ **Product**
 $f_1 \in O(g_1)$ and $f_2 \in O(g_2) \Rightarrow f_1 f_2 \in O(g_1 g_2)$
- ▶ **Sum**
 $f_1 \in O(g_1)$ and $f_2 \in O(g_2) \Rightarrow f_1 + f_2 \in O(g_1 + g_2)$
- ▶ **Multiplication with a constant**
 $k > 0$ and $f \in O(g) \Rightarrow kf \in O(g)$
 $k > 0 \Rightarrow O(kg) = O(g)$

Reason for Sufficiency of Highest-order Term

Example: $5n^3 + 2n \in O(n^3)$

- ▶ Due to rule for multiplication with a constant:
 - ▶ $5n^3 \in O(n^3)$
 - ▶ $2n \in O(n)$
- ▶ Because of $2n \in O(n)$ and $O(n) \subset O(n^3)$:
 - ▶ $2n \in O(n^3)$
- ▶ Sum rule:
 - ▶ $5n^3 + 2n \in O(n^3 + n^3)$
- ▶ Multiplication with a constant (for a class):
 - ▶ $5n^3 + 2n \in O(n^3)$

A8.3 Summary

Summary

- ▶ With **asymptotic notation**, we refer to classes of functions that **grow no faster/no slower/... than a function g** .
 - ▶ $O(g)$: Growth no faster than g .
 - ▶ $\Theta(g)$: Growth asymptotically as fast as g .