# Algorithms and Data Structures

## A7. Runtime Analysis: Bottom-Up Merge Sort

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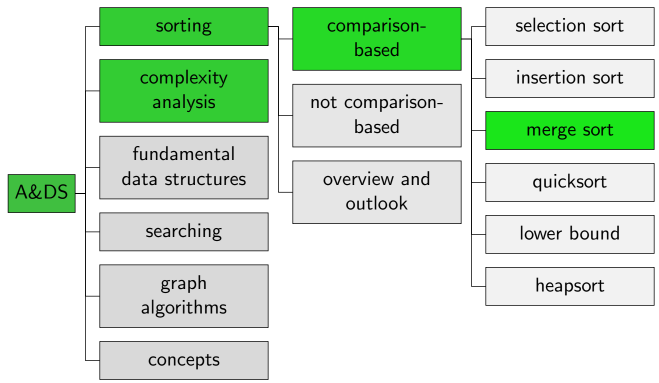
# A7.1 Runtime Analysis: Bottom-Up Merge Sort





## Content of the Course



## Merge Step

```
     def merge(array, tmp, lo, mid, hi):
c1 |     i = lo
   |     j = mid + 1
         for k in range(lo, hi + 1):  # k = lo,...,hi
   |         if j > hi or (i <= mid and array[i] <= array[j]):
   |             tmp[k] = array[i]
c2 |             i += 1
   |         else:
   |             tmp[k] = array[j]
   |             j += 1
         for k in range(lo, hi + 1):  # k = lo,...,hi
c3 |         array[k] = tmp[k]
```

We analyze the running time for $m := \text{hi} - \text{lo} + 1$
(number of elements that should be merged).





## Merge Step: Analysis

$$
\begin{aligned}
T(m) &= c_1 + c_2 m + c_3 m \\
&\geq (c_2 + c_3) m
\end{aligned}
$$

For $m \geq 1$:

$$
\begin{aligned}
T(m) &= c_1 + c_2 m + c_3 m \\
&\leq c_1 m + c_2 m + c_3 m \\
&= (c_1 + c_2 + c_3) m
\end{aligned}
$$

**Theorem**

*The merge step has linear running time, i.e., there are constants $c, c', n_0 > 0$ such that for all $n \geq n_0$: $cn \leq T(n) \leq c'n$.*





## Bottom-Up Merge Sort

```
1  def sort(array):
2      n = len(array)
3      tmp = list(array)
4      length = 1
5      while length < n:
6          lo = 0
7          while lo < n - length:
8              mid = lo + length - 1
9              hi = min(lo + 2 * length - 1, n - 1)
10             merge(array, tmp, lo, mid, hi)
11             lo += 2 * length
12         length *= 2
```

We use the following constants in the analysis:

- $c_1$ lines 2–4
- $c_2$ lines 6 and 12
- $c_3$ lines 8,9,11

Assumption: merge requires $c_4$(hi-lo+1) operations.





## Bottom-Up Merge Sort: Analysis I

Assumption: $n = 2^k$ for some $k \in \mathbb{N}_{>0}$

Iterations of the outer loop ($m$ for hi-lo+1):

- Iteration 1: $n/2$ times inner loop with merge for $m = 2$
  $c_2 + \frac{n}{2}(c_3 + 2c_4) = c_2 + \frac{1}{2}c_3 n + c_4 n$
- Iteration 2: $n/4$ times inner loop with merge for $m = 4$
  $c_2 + \frac{n}{4}(c_3 + 4c_4) = c_2 + \frac{1}{4}c_3 n + c_4 n$
- ...
- Outer loop terminates after last iteration $\ell$.
- Iteration $\ell$: $n/n = 1$ time inner loop with merge for $m = n$
  $c_2 + \frac{n}{n}(c_3 + nc_4) = c_2 + c_3 + c_4 n$

Total $T(n) \leq c_1 + \ell(c_2 + c_3 n + c_4 n) \leq \ell(c_1 + c_2 + c_3 + c_4)n$

## Bottom-Up Merge Sort: Analysis II

What is the value of $\ell$?

- In iteration $i$ we have $m = 2^i$ for the merge step.
- In iteration $\ell$ we have $m = 2^\ell = n$ for the merge step.
- Since $n = 2^k$ we have $\ell = k = \log_2 n$.

With $c := c_1 + c_2 + c_3 + c_4$ we get $T(n) \leq cn \log_2 n$.





## Bottom-Up Merge Sort: Analysis III

What if $n$ is not a power of two, so $2^{k-1} < n < 2^k$?

- Nevertheless $k$ iterations of the outer loop.
- Inner loop does not perform more operations.
- $T(n) \leq cnk = cn(\lfloor \log_2 n \rfloor + 1) \leq 2cn \log_2 n$ (for $k > 2$)





## Bottom-Up Merge Sort: Analysis IV

Analogous argument possible for lower bound.
$\rightarrow$ Exercises

**Theorem**
*Bottom-up merge sort has linearithmic running time, i.e. there are constants $c, c', n_0 > 0$, such that for all $n \geq n_0$: $cn \log_2 n \leq T(n) \leq c' n \log_2 n$.*





## Linearithmic Running Time

Linearithmic running time $n \log_2 n$:
$\rightarrow$ twice as large input, slightly more than twice the running time

What does this mean in practice?

- Assumption: $c = 1$, one operation takes on average $10^{-8}$ sec.
- With 1000 elements, we wait $10^{-8} \cdot 10^3 \log_2(10^3) \approx$ 0.0001 seconds.
- With 10 thousand elements $\approx$ 0.0013 seconds.
- With 100 thousand elements $\approx$ 0.017 seconds.
- With 1 million elements $\approx$ 0.2 seconds.
- With 1 billion elements $\approx$ 299 seconds.

Running time $n \log_2 n$ not much worse than linear running time

## Merge Sort with Cost Model I

Key comparisons

- Only in `merge`.
- Merging two ranges of length $m$ and $n$ requires in the best case $\min(n, m)$ and in the worst case $n + m - 1$ comparisons.
- With two ranges of roughly equal length, this is a linear number of comparisons, i.e., there are $c, c' > 0$ such that the number of comparisons is between $cn$ and $c'n$.

→ Number of key comparisons that is performed for sorting the entire input sequence is linearithmic in the length of the sequence (analogously to the runtime analysis).





## Merge Sort with Cost Model II

Movements of elements

- Only in `merge`.
- $2n$ movements for sequence of length $n$.
- Total for merge sort linearithmic (analogously to key comparisons).





# A7.2 Summary





## Summary

- Merge sort has linearithmic running time, key comparisons and movements of elements.