Algorithms and Data Structures

A5. Runtime Analysis: Introduction and Selection Sort

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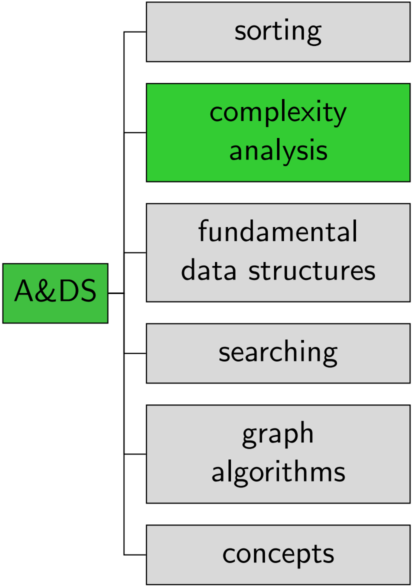
Runtime Analysis in General

Content of the Course



Exact Runtime Analysis Unrealistic

- Would be nice: formula that determines for a specific input how long the computation will take.
- Exact runtime prediction is hard because of too many influencing factors.
  - Speed and architecture of the computer
  - Programming language
  - Compiler version
  - Current load (what else is running?)
  - Caching behavior

We neither can nor want to consider all this in a formula.

Runtime Analysis: 1st Simplification

Don't measure time but count operations

What is an operation?

- Ideally: one line of machine code or – even more precisely – one processor cycle
- Instead: constant-time operations
  - Constant time: running time independent of input.
  - Ignore runtime differences of different operations.
  - E.g. addition, assignments, branching, function call.
  - Roughly: operation = one line of code.
  - But: also consider what's behind it
    e.g. steps inside the called function.

Running time roughly proportional to the number of operations

Runtime Analysis: 2nd Simplification

Don’t count exactly but use bounds!

- Mostly considering upper bounds
  How many steps does it take at most?
- Sometimes also lower bound
  How many steps are at least executed?

"‘running time"’ for bound on number of executed operations

Runtime Analysis: 3rd Simplification

Bounds only relative to the input size

- $T(n)$: running time for input of size n
- For adaptive algorithms we distinguish
  - Best case
    running time for best possible input of size n
  - Worst case
    running time for worst possible input of size n
  - Average case
    average running time over all inputs of size n

Cost Models

Sometimes: analysis wrt. cost model

- Identify fundamental operations for the algorithm class e.g. for sorting algorithms.
  - Key comparison
  - Swap of two elements or movement of an element
- Analyze number of these operations.

Example from C++ Reference

function template

std::sort <algorithm>

```
default (1)  template <class RandomAccessIterator>
               void sort (RandomAccessIterator first, RandomAccessIterator last);
custom (2)   template <class RandomAccessIterator, class Compare>
               void sort (RandomAccessIterator first, RandomAccessIterator last, Compare comp);
```

Sort elements in range

Sorts the elements in the range `[first,last)` into ascending order.

The elements are compared using `operator<` for the first version, and *comp* for the second.

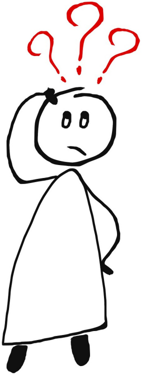
Equivalent elements are not guaranteed to keep their original relative order (see `stable_sort`).

Complexity

On average, linearithmic in the distance between *first* and *last*: Performs approximately $N*\log_2(N)$ (where N is this distance) comparisons of elements, and up to that many element swaps (or moves).

http://www.cplusplus.com/reference/algorithm/sort/

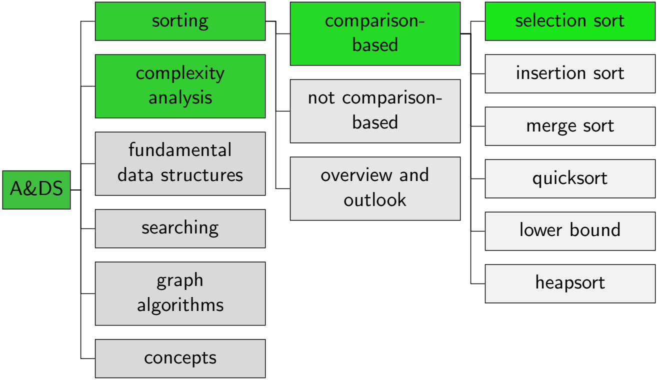
Questions



Questions?

Example: Selection Sort

Content of the Course



Selection Sort: Algorithm

```
def selection_sort(array):
    n = len(array)
    for i in range(n - 1):  # i = 0, ..., n-2
        # find index of minimum element at positions i, ..., n-1
        min_index = i
        for j in range(i + 1, n):  # j = i+1, ..., n-1
            if array[j] < array[min_index]:
                min_index = j
        # swap element at position i with minimum element
        array[i], array[min_index] = array[min_index], array[i]
```

Selection Sort with Cost Model

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def selection_sort(array):
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            if array[j] < array[min_index]:
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        # swap element at position i with minimum element
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```

On an input of size n, how often does the algorithm swap two elements?

Selection Sort with Cost Model

```python
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→ n-1 swaps of two elements ("linear")

Selection Sort with Cost Model

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```

→ n-1 swaps of two elements ("linear")
→ $\binom{n}{2} = \frac{1}{2}n(n-1)$ key comparisons ("quadratic")

Selection Sort: Analysis I

We show: $T(n) \leq c' \cdot n^2$ for $n \geq 1$ and some constant c'

- Outer loop (3-10) and inner loop (6-8)
- Number of operations for each iteration of the outer loop:

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i	# operations
0	$a(n-1)+b$
1	$a(n-2)+b$
	$\ldots$

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i	# operations
0	$a(n-1)+b$
1	$a(n-2)+b$
	...
n-2	$a \cdot 1 + b$

- Total: $T(n) = \sum_{i=0}^{n-2}(a(n-(i+1))+b)$

Selection Sort: Analysis II

$$T(n) = \sum_{i=0}^{n-2} (a(n-(i+1)) + b)$$

Selection Sort: Analysis II

$$\begin{aligned}T(n) &= \sum_{i=0}^{n-2}(a(n-(i+1))+b) \\ &= \sum_{i=1}^{n-1}(a(n-i)+b)\end{aligned}$$

Selection Sort: Analysis II

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{n-2}(a(n-(i+1))+b) \\
&= \sum_{i=1}^{n-1}(a(n-i)+b) \\
&= a\sum_{i=1}^{n-1}(n-i)+b(n-1)
\end{aligned}
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&= a\sum_{i=1}^{n-1}(n-i)+b(n-1) \\
&= \frac{1}{2}a(n-1)n+b(n-1)
\end{aligned}
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Selection Sort: Analysis II

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T(n) &= \sum_{i=0}^{n-2} (a(n-(i+1)) + b) \\
&= \sum_{i=1}^{n-1} (a(n-i) + b) \\
&= a\sum_{i=1}^{n-1} (n-i) + b(n-1) \\
&= \frac{1}{2}a(n-1)n + b(n-1) \\
&\leq \frac{1}{2}an^2 + b(n-1)
\end{aligned}$$

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&\leq \frac{1}{2}an^2+b(n-1)n
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&\leq \frac{1}{2}an^2+b(n-1)n \\
&\leq \frac{1}{2}an^2+bn^2 \qquad = (\frac{1}{2}a+b)n^2
\end{aligned}
$$

$\Rightarrow$ with $c' = (\frac{1}{2}a+b)$ it holds for $n \geq 1$ that $T(n) \leq c' \cdot n^2$

Selection Sort: Analysis III

Too generous bound?

We show for $n \geq 2$: $T(n) \geq c \cdot n^2$ for some constant c

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$$
\begin{aligned}
T(n) = \cdots &= \frac{1}{2}a(n-1)n + b(n-1) \\
&\geq \frac{1}{2}a(n-1)n \\
&\geq \frac{1}{4}an^2 \qquad\qquad (n-1 \geq \frac{1}{2}n \text{ for } n \geq 2)
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$\Rightarrow$ with $c = \frac{1}{4}a$ it holds for $n \geq 2$ that $T(n) \geq c \cdot n^2$

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$$

$\Rightarrow$ with $c = \frac{1}{4}a$ it holds for $n \geq 2$ that $T(n) \geq c \cdot n^2$

Theorem

Selection sort has quadratic running time, i.e., there are constants $c > 0, c' > 0, n_0 > 0$ such that for $n \geq n_0$: $cn^2 \leq T(n) \leq c'n^2$.

Selection Sort: Analysis IV

Quadratic running time: twice as large input, fourfold running time

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- With 10 thousand elements, we wait $10^{-8} \cdot (10^4)^2 =$ 1 second.
- With 100 thousand elements $10^{-8} \cdot (10^5)^2 =$ 100 seconds.

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- With 100 thousand elements $10^{-8} \cdot (10^5)^2 = 100$ seconds.
- With 1 million elements $10^{-8} \cdot (10^6)^2$ seconds $= 2.77$ hours.

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- With 100 thousand elements $10^{-8} \cdot (10^5)^2 = 100$ seconds.
- With 1 million elements $10^{-8} \cdot (10^6)^2$ seconds $= 2.77$ hours.
- With 1 billion elements $10^{-8} \cdot (10^9)^2$ seconds $= 317$ years.
  1 billion numbers with 4 bytes/number are "‘only"’ 4 GB.

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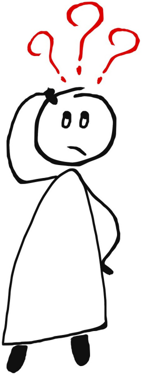
Quadratic running time problematic for large inputs

Questions



Questions?

Summary

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- Runtime analysis considers bounds on the number of executed operations.
  - We don't count exactly.
  - We ignore how long each operation actually takes.
  - Running time should be roughly proportional to the number of operations.
- Selection sort has quadratic running time and performs a linear number of swaps and a quadratic number of key comparisons.