# Theory of Computer Science

## D1. Nondeterministic Algorithms, P and NP

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## Overview: Course

contents of this course:

A. background ✓
  - ▷ mathematical foundations and proof techniques

B. automata theory and formal languages ✓
  - ▷ What is a computation?

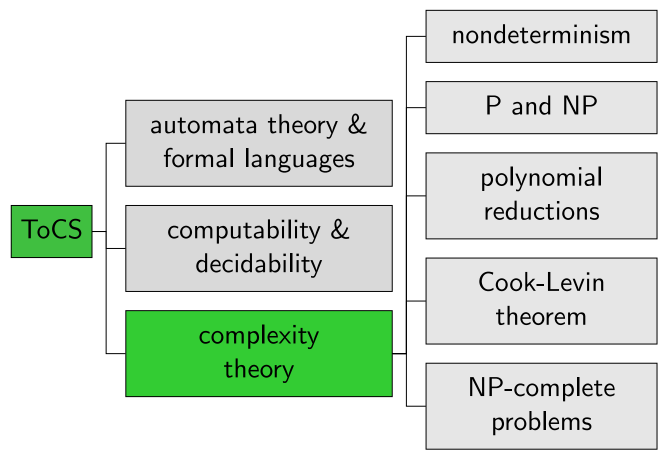
C. Turing computability ✓
  - ▷ What can be computed at all?

D. complexity theory
  - ▷ What can be computed efficiently?

E. more computability theory
  - ▷ Other models of computability



## Content of the Course



# D1.1 Motivation





## A Scenario (1)

**Example Scenario**

- You are a programmer at a logistics company.
- Your boss gives you the task of developing a program to optimize the route of a delivery truck:
  - The truck begins its route at the company depot.
  - It has to visit 50 stops.
  - You know the distances between all relevant locations (stops and depot).
  - Your program should compute a tour visiting all stops and returning to the depot on a shortest route.





## A Scenario (2)

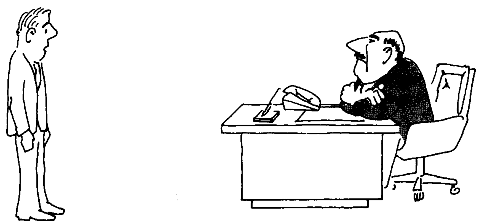
**Example Scenario (ctd.)**

- You work on the problem for weeks, but you do not manage to complete the task.
- All of your attempted programs
  - compute routes that are possibly suboptimal, or
  - do not terminate in reasonable time (say: within a month).
- What do you say to your boss?





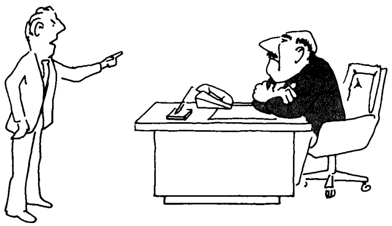
## What You Don't Want to Say



"I can't find an efficient algorithm,
I guess I'm just too dumb."

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 2

## What You Would Like to Say



"I can't find an efficient algorithm,
because no such algorithm is possible!"

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 2





## What Complexity Theory Allows You to Say

"I can't find an efficient algorithm,
but neither can all these famous people."

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 3





## Why Complexity Theory?

**Complexity Theory**

Complexity theory tells us which problems can be solved quickly ("simple problems") and which ones cannot ("hard problems").

- This is useful in practice because simple and hard problems require different techniques to solve.
- If we can show that a problem is hard we do not need to waste our time with the (futile) search for a "simple" algorithm.





## Test Your Intuition! (1)

- The following slide lists some graph problems.
- The input is always a directed graph $G = \langle V, E \rangle$.
- How difficult are the problems in your opinion?
- Sort the problems
  from easiest (= requires least amount of time to solve)
  to hardest (= requires most time to solve)
- no justification necessary, just follow your intuition!
- anonymous and not graded

## Test Your Intuition! (2)

1. Find a simple path (= without cycle) from $u \in V$ to $v \in V$ with minimal length.
2. Find a simple path (= without cycle) from $u \in V$ to $v \in V$ with maximal length.
3. Determine whether $G$ is strongly connected (every node is reachable from every other node).
4. Find a cycle (non-empty path from $u$ to $u$ for any $u \in V$; multiple visits of nodes are allowed).
5. Find a cycle that visits all nodes.
6. Find a cycle that visits a given node $u$.
7. Find a path that visits all nodes without repeating a node.
8. Find a path that uses all edges without repeating an edge.





# D1.2 How to Measure Running Time?





## How to Measure Running Time?

- Time complexity is a way to measure how much time it takes to solve a problem.
- How can we define such a measure appropriately?





## Example Statements about Running Time

Example statements about running time:

- "Running `sort /usr/share/dict/words` on the computer `dakar` takes 0.035 seconds."
- "With a 1 MiB input file, `sort` takes at most 1 second on a modern computer."
- "Quicksort is faster than sorting by insertion."
- "Sorting by insertion is slow."

$\rightsquigarrow$ Very different statements with different pros and cons.

## Precise Statements vs. General Statements

**Example Statement about Running Time**
"Running `sort /usr/share/dict/words` on the computer `dakar` takes 0.035 seconds."

advantage: very precise

disadvantage: not general

- input-specific:
  What if we want to sort other files?
- machine-specific:
  What happens on a different computer?
- even situation-specific:
  Will we get the same result tomorrow that we got today?





## General Statements about Running Time

In this course we want to make general statements about running time. We accomplish this in three ways:

**1. General Inputs**
Instead of concrete inputs, we talk about general types of input:

- Example: running time to sort an input of size $n$ in the worst case
- Example: running time to sort an input of size $n$ in the average case

here: running time for input size $n$ in the worst case





## General Statements about Running Time

In this course we want to make general statements about running time. We accomplish this in three ways:

**2. Ignoring Details**
Instead of exact formulas for the running time we specify the order of magnitude:

- Example: instead of saying that we need time $\lceil 1.2n \log n \rceil - 4n + 100$, we say that we need time $O(n \log n)$.
- Example: instead of saying that we need time $O(n \log n)$, $O(n^2)$ or $O(n^4)$, we say that we need polynomial time.

here: What can be computed in polynomial time?





## General Statements about Running Time

In this course we want to make general statements about running time. We accomplish this in three ways:

**3. Abstract Cost Measures**
Instead of the running time on a concrete computer we consider a more abstract cost measure:

- Example: count the number of executed machine code statements
- Example: count the number of executed Java byte code statements
- Example: count the number of element comparisons of a sorting algorithms

here: count the computation steps of a Turing machine (polynomially equivalent to other measures)

# D1.3 Decision Problems





## Decision Problems

- As before, we simplify our investigation by restricting our attention to decision problems.
- More complex computational problems can be solved with multiple queries for an appropriately defined decision problem ("playing 20 questions").
- Formally, decision problems are languages (as before), but we use an informal "given" / "question" notation where possible.





## Example: Decision vs. General Problem (1)

**Definition (Hamilton Cycle)**

Let $G = \langle V, E \rangle$ be a (directed or undirected) graph.

A Hamilton cycle of $G$ is a sequence of vertices in $V$, $\pi = \langle v_0, \dots, v_n \rangle$, with the following properties:

- $\pi$ is a path: there is an edge from $v_i$ to $v_{i+1}$ for all $0 \leq i < n$
- $\pi$ is a cycle: $v_0 = v_n$
- $\pi$ is simple: $v_i \neq v_j$ for all $i \neq j$ with $i, j < n$
- $\pi$ is Hamiltonian: all nodes of $V$ are included in $\pi$





## Example: Decision vs. General Problem (2)

**Example (Hamilton Cycles in Directed Graphs)**

$\mathcal{P}$: general problem DirHamiltonCycleGen

- Input: directed graph $G = \langle V, E \rangle$
- Output: a Hamilton cycle of $G$ or a message that none exists

$\mathcal{D}$: decision problem DirHamiltonCycle

- Given: directed graph $G = \langle V, E \rangle$
- Question: Does $G$ contain a Hamilton cycle?

These problems are polynomially equivalent: from a polynomial algorithm for one of the problems one can construct a polynomial algorithm for the other problem. (Without proof.)

## Algorithms for Decision Problems

Algorithms for decision problems:

- Where possible, we specify algorithms for decision problems in pseudo-code.
- Since they are only yes/no questions, we do not have to return a general result.
- Instead we use the statements
  - **ACCEPT** to accept the given input ("yes" answer) and
  - **REJECT** to reject it ("no" answer).
- Where we must be more formal, we use Turing machines and the notion of accepting from chapter B11.



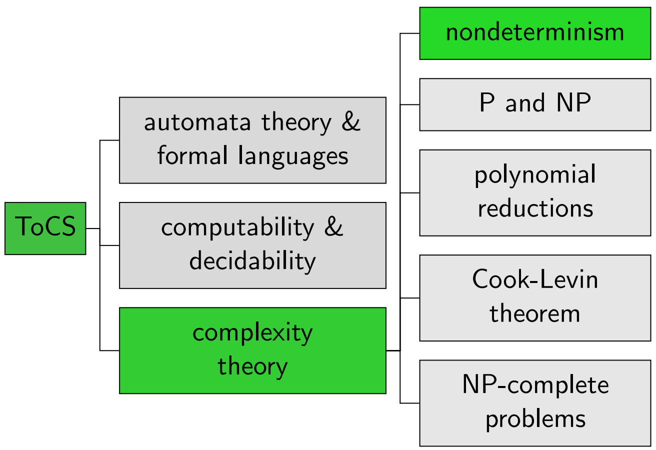


# D1.4 Nondeterminism





## Content of the Course








## Nondeterminism

- To develop complexity theory, we need the algorithmic concept of nondeterminism.
- already known for Turing machines (⇝ chapter B11):
  - An NTM can have more than one possible successor configuration for a given configuration.
  - Input $x$ is accepted if there is at least one possible computation (configuration sequence) that leads to the accept state.
- Here we analogously introduce nondeterminism for pseudo-code.

# Nondeterministic Algorithms

nondeterministic algorithms:

- All constructs of deterministic algorithms are also allowed in nondeterministic algorithms: **IF**, **WHILE**, etc.
- Additionally, there is a nondeterministic assignment:
  **GUESS** $x_i \in \{0, 1\}$

  where $x_i$ is a program variable.





# Nondeterministic Algorithms: Acceptance

- Meaning of **GUESS** $x_i \in \{0, 1\}$:
  $x_i$ is assigned either the value 0 or the value 1.
- This implies that the behavior of the program on a given input is no longer uniquely defined: there are multiple possible execution paths.
- The program accepts a given input if at least one execution path leads to an **ACCEPT** statement.
- Otherwise, the input is rejected.

Note: asymmetry between accepting and rejecting!
(cf. Turing-recognizability)





# More Complex GUESS Statements

- We will also guess more than one bit at a time:
  **GUESS** $x \in \{1, 2, \dots, n\}$

  or more generally
  **GUESS** $x \in S$

  for a finite set $S$.
- These are abbreviations and can be split into $\lceil \log_2 n \rceil$ (or $\lceil \log_2 |S| \rceil$) "atomic" **GUESS** statements.





# Example: Nondeterministic Algorithms (1)

Example (DirHamiltonCycle)

input: directed graph $G = \langle V, E \rangle$

*start* := an arbitrary node from $V$
*current* := *start*
*remaining* := $V \setminus \{start\}$
**WHILE** $remaining \neq \emptyset$:
    **GUESS** $next \in remaining$
    **IF** $\langle current, next \rangle \notin E$:
        **REJECT**
    *remaining* := $remaining \setminus \{next\}$
    *current* := *next*
**IF** $\langle current, start \rangle \in E$:
    **ACCEPT**
**ELSE**:
    **REJECT**

## Example: Nondeterministic Algorithms (2)

- With appropriate data structures, this algorithm solves the problem in $O(n \log n)$ program steps, where $n = |V| + |E|$ is the size of the input.
- How many steps would a deterministic algorithm need?





## Guess and Check

- The DirHamiltonCycle example illustrates a general design principle for nondeterministic algorithms:

  guess and check
- In general, nondeterministic algorithms can solve a problem by first guessing a "solution" and then verifying that it is indeed a solution. (In the example, these two steps are interleaved.)
- If solutions to a problem can be efficiently verified, then the problem can also be efficiently solved if nondeterminism may be used.





## The Power of Nondeterminism

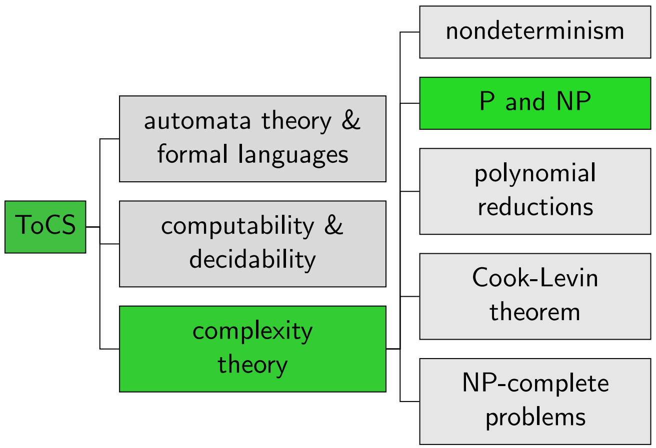
- Nondeterministic algorithms are very powerful because they can "guess" the "correct" computation step.
- Or, interpreted differently: they go through many possible computations "in parallel", and it suffices if one of them is successful.
- Can they solve problems efficiently (in polynomial time) which deterministic algorithms cannot solve efficiently?
- This is the big question!





# D1.5 P and NP

## Content of the Course








## Impact of Nondeterminism?

- We earlier established that deterministic and nondeterministic Turing machines recognize the same class of languages.
  $\rightarrow$ For this aspect, nondeterminism did not make a difference.
- Now we consider what decision problems can be solved in polynomial time.
- Does it make a difference whether we allow nondeterminism?

This is the famous P vs. NP question!





## Running Time of a Deterministic Turing Machine

**Definition (Running Time of a DTM)**

Let $M$ be a DTM that halts on all inputs. The running time or time complexity of $M$ is the function $f : \mathbb{N} \rightarrow \mathbb{N}$, where $f(n)$ is the maximum number of steps that $M$ uses on any input of length $n$.

We say that

- $M$ runs in time $f$ and that
- $M$ is an $f$ time Turing machine.





## Big-O

**Definition (Big-O)**

Let $f$ and $g$ be functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$.

We say that $f \in O(g)$ if positive integers $c$ and $n_0$ exist such that for every integer $n \geq n_0$

$$f(n) \leq cg(n).$$

## Complexity Class P

### Definition (Time Complexity Class TIME)

Let $t : \mathbb{N} \to \mathbb{R}^+$ be a function.

Define the time complexity class TIME(t(n)) to be the collection of all languages that are decidable by an $O(t)$ time Turing machine.

### Definition (P)

P is the class of languages that are decidable in polynomial time by a deterministic single-tape Turing machine. In other words,

$$P = \bigcup_k \mathrm{TIME}(n^k).$$





## Running Time of a Nondeterministic Turing Machine

### Definition (Running Time of a NTM)

Let $M$ be a NTM that is a decider, i. e. all its computation branches halt on all inputs.

The running time or time complexity of $M$ if the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum number of steps that $M$ uses on any branch of its computation on any input of length $n$.





## Complexity Class NP

### Definition (Time Complexity Class NTIME)

Let $t : \mathbb{N} \to \mathbb{R}^+$ be a function.

Define the time complexity class NTIME(t(n)) to be the collection of all languages that are decidable by an $O(t)$ time nondeterministic Turing machine.

### Definition (NP)

NP is the class of languages that are decidable in polynomial time by a nondeterministic single-tape Turing machine. In other words,

$$NP = \bigcup_k \mathrm{NTIME}(n^k).$$





## P and NP: Remarks

- Sets of languages like P and NP that are defined in terms of computation time of TMs (or other computation models) are called complexity classes.
- We know that $P \subseteq NP$. (Why?)
- Whether the converse is also true is an open question: this is the famous P-NP problem.

## Example: DirHamiltonCycle $\in$ NP

**Example (DirHamiltonCycle $\in$ NP)**
The nondeterministic algorithm of the previous section solves the problem and can be implemented on an NTM in polynomial time.

- Is DirHamiltonCycle $\in$ P also true?
- The answer is unknown.
- So far, only exponential deterministic algorithms for the problem are known.





## Simulation of NTMs with DTMs

- Unlike DTMs, NTMs are not a realistic computation model: they cannot be directly implemented on computers.
- But NTMs can be simulated by systematically trying all computation paths, e. g., with a breadth-first search.

More specifically:

- Let $M$ be an NTM that decides language $L$ in time $f$, where $f(n) \geq n$ for all $n \in \mathbb{N}_0$.
- Then we can specify a DTM $M'$ that decides $L$ in time $f'$, where $f'(n) = 2^{O(f(n))}$.
- without proof
  (cf. "Introduction to the Theory of Computation" by Michael Sipser (3rd edition), Theorem 7.11)





## Summary (1)

- Complexity theory deals with the question which problems can be solved efficiently and which ones cannot.
- here: focus on what can be computed in polynomial time
- To formalize this, we use Turing machines, but other formalisms are polynomially equivalent.
- We consider decision problems, but the results often directly transfer to general computational problems.





## Summary (2)

important concept: nondeterminism

- Nondeterministic algorithms can "guess", i. e., perform multiple computations "at the same time".
- An input receives a "yes" answer if at least one computation path accepts it.
- in NTMs: with nondeterministic transitions ($\delta(q, a)$ contains multiple elements)
- in pseudo-code: with **GUESS** statements

# Summary (3)

- P: languages decidable by DTMs in polynomial time
- NP: languages decidable by NTMs in polynomial time
- P $\subseteq$ NP but it is an open question whether P $=$ NP.