

Theory of Computer Science

C4. Reductions

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Introduction

What We Achieved So Far: Discussion

- We already know a concrete undecidable problem.
→ halting problem
- We will see that we can derive **further** undecidability results from the undecidability of the halting problem.
- The central notion for this is **reducing** one problem to another problem.

Illustration

```
def is_odd(some_number):  
    n = some_number + 1  
    return is_even(n)
```

- Decides whether a given number is odd based on...
- an algorithm that determines whether a number is even.

Reduction: Idea (slido)

Assume that you have an algorithm that solves problem A relying on a hypothetical algorithm for problem B.

```
def is_in_A(input_A):  
    input_B = <compute suitable instance based on input_A>  
    return is_in_B(input_B)
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What (if anything) can you conclude

- 1 if there indeed is an algorithm for problem A ?
- 2 if there indeed is an algorithm for problem B ?
- 3 if problem A is undecidable?
- 4 if problem B is undecidable?



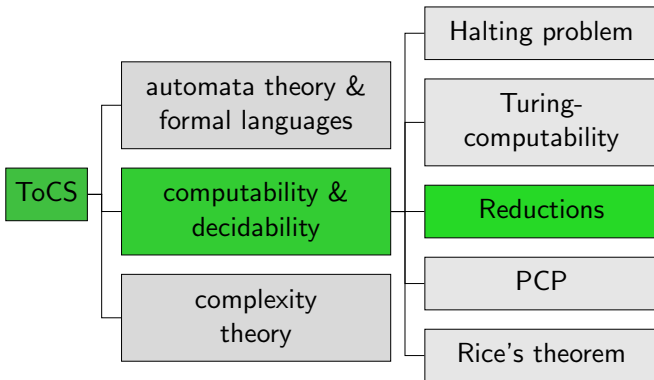
Questions



Questions?

Reduction

Content of the Course



Reduction: Definition

Definition (Reduction)

Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be languages, and let $f : \Sigma^* \rightarrow \Gamma^*$ be a total and computable function such that for all $x \in \Sigma^*$:

$$x \in A \quad \text{if and only if} \quad f(x) \in B.$$

Then we say that A can be **reduced to B** (in symbols: $A \leq B$), and f is called a **reduction from A to B** .

Reduction Property

Theorem (Reductions vs. Turing-recognizability/Decidability)

Let A and B be languages with $A \leq B$. Then:

- 1 If B is decidable, then A is decidable.
- 2 If B is Turing-recognizable, then A is Turing-recognizable.
- 3 If A is not decidable, then B is not decidable.
- 4 If A is not Turing-recognizable, then B is not Turing-recognizable.

↪ In the following, we use 3. to show undecidability for further problems.

Reduction Property: Proof

Proof.

for 1.: If B is decidable then there is a DTM M_B that decides B . The following algorithm decides A using reduction f from A to B .

On input x :

- 1 $y := f(x)$
- 2 Simulate M_B on input y . This simulation terminates.
- 3 If M_B accepted y , accept. Otherwise reject.

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for 2.: identical to (1), only that M_B only recognizes B and therefore the simulation does not necessarily terminate if $y \notin B$. Since $y \notin B$ iff $x \notin A$, the procedure still recognizes A .

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for 3./4.: contrapositions of 1./2. \rightsquigarrow logically equivalent □

Reductions are Preorders

Theorem (Reductions are Preorders)

The relation " \leq " is a preorder:

- 1 For all languages A :
 $A \leq A$ (*reflexivity*)
- 2 For all languages A, B, C :
If $A \leq B$ and $B \leq C$, then $A \leq C$ (*transitivity*)

Reductions are Preorders: Proof

Proof.

for 1.: The function $f(x) = x$ is a reduction from A to A because it is total and computable and $x \in A$ iff $f(x) \in A$.

for 2.: \rightsquigarrow exercises



Questions



Questions?

Halting Problem on Empty Tape

Example

As an example

- we will consider problem H_0 , a variant of the halting problem,
- ... and show that it is undecidable
- ... reducing H to H_0 .

Reminder: Halting Problem

Definition (Halting Problem)

The **halting problem** is the language

$$H = \{w\#x \in \{0, 1, \#\}^* \mid w, x \in \{0, 1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

Halting Problem on Empty Tape (1)

Definition (Halting Problem on the Empty Tape)

The **halting problem on the empty tape** is the language

$$H_0 = \{w \in \{0, 1\}^* \mid M_w \text{ started on } \varepsilon \text{ terminates}\}.$$

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The **halting problem on the empty tape** is the language

$$H_0 = \{w \in \{0, 1\}^* \mid M_w \text{ started on } \varepsilon \text{ terminates}\}.$$

Note: H_0 is Turing-recognizable. (Why?)

Halting Problem on Empty Tape (2)

Theorem (Undecidability of Halting Problem on Empty Tape)

The halting problem on the empty tape is undecidable.

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Proof.

We show $H \leq H_0$.

Consider the function $f : \{0, 1, \#\}^* \rightarrow \{0, 1\}^*$

that computes the word $f(z)$ for a given $z \in \{0, 1, \#\}^*$ as follows:

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- Test if z has the form $w\#x$ with $w, x \in \{0, 1\}^*$.
- If not, return any word that is not in H_0
(e. g., encoding of a TM that instantly starts an endless loop).
- If yes, split z into w and x .

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- If yes, split z into w and x .
- Decode w to a TM M_2 .

Halting Problem on Empty Tape (3)

Proof (continued).

- Construct a TM M_1 that behaves as follows:
 - If the input is empty: write x onto the tape and move the head to the first symbol of x (if $x \neq \varepsilon$); then stop
 - otherwise, stop immediately

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- Construct TM M that first runs M_1 and then M_2 .
→ M started on empty tape simulates M_2 on input x .

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f is total and (with some effort) computable. Also:

$z \in H$ iff $z = w\#x$ and M_w run on x terminates
iff $M_{f(z)}$ started on empty tape terminates
iff $f(z) \in H_0$

$\rightsquigarrow H \leq H_0 \rightsquigarrow H_0$ undecidable



Questions



Questions?

Summary

Summary

- **reductions:** “embedding” a problem as a special case of another problem
- important method for proving undecidability:
reduce from a known undecidable problem to a new problem