Theory of Computer Science

C2. The Halting Problem

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Turing-recognizable vs. decidable

Plan for this Chapter

- We will first revisit the notions Turing-recognizable and Turing-decidable and identify a connection between the two concepts.
- Then we will get to know an important undecidable problem, the halting problem.
- We show that it is Turing-recognizable...
- ... but not Turing-decidable.
- From these results we can conclude that there are languages that are not Turing-recognizable.
- Some of the postponed results on the closure and decidability properties of type 0 languages are direct implications of our findings.

Reminder: Turing-recognizable and Turing-decidable

Definition (Turing-recognizable Language)

We call a language Turing-recognizable if some deterministic Turing machine recognizes it.

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We call a language Turing-recognizable if some deterministic Turing machine recognizes it.

A Turing machine that halts on all inputs (entering q_{reject} or q_{accept}) is a decider. A decider that recognizes some language also is said to decide the language.

Definition (Turing-decidable Language)

We call a language Turing-decidable (or decidable) if some deterministic Turing machine decides it.

Intuition

Are these two definitions meaningfully different?

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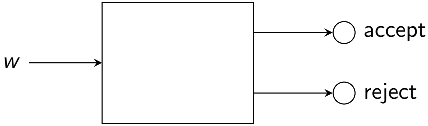
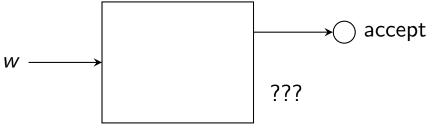


Turing-recognizable



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Case 1: $w \in L$

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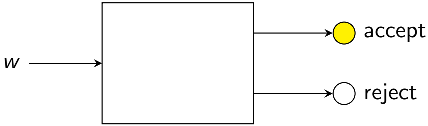

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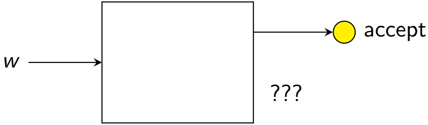

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Case 2: $w \notin L$

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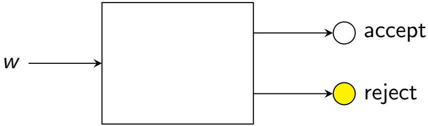

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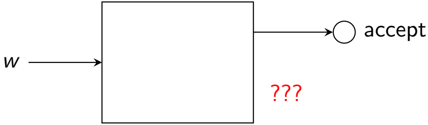

Connection Turing-recognizable and Turing-decidable (1)

Reminder: For language L, we write $\bar{L}$ do denote its complement.

Theorem (Decidable vs. Turing-recognizable)

A language L is decidable iff both L and $\bar{L}$ are Turing-recognizable.

Proof.

$(\Rightarrow)$: obvious (Why?) . . .

Connection Turing-recognizable and Turing-decidable (2)

Proof (continued).

($\Leftarrow$): Let M_L be a DTM that recognizes L,
and let $M_{\bar{L}}$ be a DTM that recognizes $\bar{L}$.

The following algorithm decides L:

On a given input word w proceed as follows:
FOR $s := 1, 2, 3, \dots$:
  IF M_L stops on w in s steps in the accept state:
    ACCEPT
  IF $M_{\bar{L}}$ stops on w in s steps in the accept state:
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Connection Turing-recognizable and Turing-decidable (2)

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Why don't we first entirely simulate M_L on the input and only afterwards $M_{\bar{L}}$?

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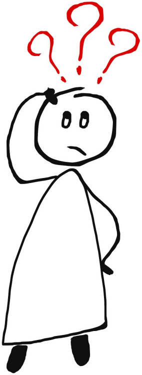
- We just do not know what is the correct version (and what is n_0 in case 2).

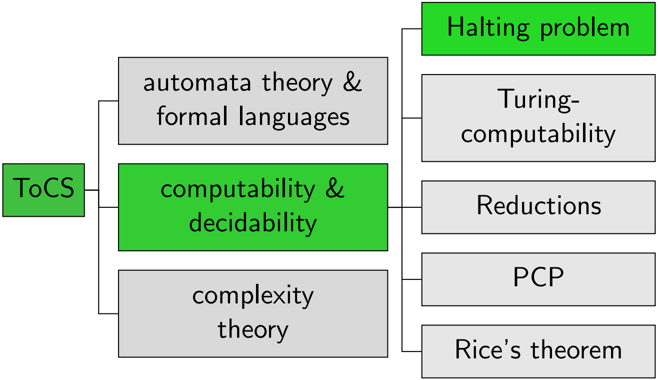
Questions



Questions?

The Halting Problem H

Content of the Course



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- To define for every $w \in \{0,1\}^*$ a corresponding TM, we use an arbitrary fixed DTM $\widehat{M}$ and define

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$$M_w = \begin{cases} M' & \text{if } w \text{ is the encoding of some DTM } M' \\ \widehat{M} & \text{otherwise} \end{cases}$$

- $M_w =$ "Turing machine encoded by w"

Halting Problem

Definition (Halting Problem)

The halting problem is the language

$$H = \{w\#x \in \{0,1,\#\}^* \mid w, x \in \{0,1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

"Does the computation of the TM encoded by w halt on input x?"
"Does a given piece of code terminate on a given input?"

The Halting Problem is Turing-recognizable

Theorem

The halting problem H is Turing-recognizable.

The following Turing machine U recognizes language H:

On input $w\#x$:

1. If the input contains more than one $\#$ then reject.
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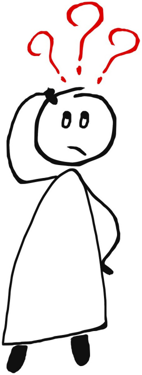
U is an example of a so-called *universal Turing machine*
which can simulate any other Turing machine
from the description of that machine.

Questions



Questions?

H is Undecidable

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- To establish the undeciability of the halting problem, we will consider a situation where we run a Turing machine/algorithm on its own encoding/source code.
- We have seen something similar in the very first lecture. . .

Uncomputable Problems?

Consider functions whose inputs are strings:

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def program_returns_true_on_input(prog_code, input_str):
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    # returns True if prog_code run on input_str returns True
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def weird_program(prog_code):
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What is the return value of `weird_program`
if we run it on its own source code?

Solution

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- Overall, we have proven that there cannot be a program with the behaviour described by the comments.
- For the undecidability of the halting problem, we will use an analogous argument, only with Turing machines instead of code and termination instead of return values.

Undecidability of the Halting Problem (1)

Theorem (Undecidability of the Halting Problem)

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Proof by contradiction: we assume that the halting problem H was decidable and derive a contradiction.

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Proof by contradiction: we assume that the halting problem H was decidable and derive a contradiction.

So assume H is decidable and let D be a DTM that decides it. ...

Undecidability of the Halting Problem (2)

Proof (continued).

Construct the following new machine M that takes a word $x \in \{0, 1\}^*$ as input:

1. Execute D on the input $x\#x$.
2. If it rejects: accept.
3. Otherwise: enter an endless loop.

Undecidability of the Halting Problem (2)

Proof (continued).

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Let w be the encoding of M. How will M behave on input w?

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M run on w stops
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iff M run on w does not stop (remember that w encodes M)

Contradiction! DTM M cannot exist.

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Contradiction! DTM M cannot exist.
$\Rightarrow$ DTM D cannot exist, thus H is not decidable. □

A Language that is not Turing-recognizable

We have the following results:

- A language L is decidable iff both L and $\bar{L}$ are Turing-recognizable.
- The halting problem H is Turing-recognizable but not decidable.

Corollary

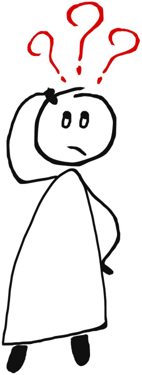
The complement $\bar{H}$ of the halting problem H is not Turing-recognizable.

Exercises

- True or false? There is a grammar that generates H.
- True or false? Not all languages are of type 0.

Justify your answers.

Questions



Questions?

Reprise: Type-0 Languages

Back to Chapter B13: Closure Properties

	Intersection	Union	Complement	Concatenation	Star
Type 3	Yes	Yes	Yes	Yes	Yes
Type 2	No	Yes	No	Yes	Yes
Type 1	Yes$^{(2)}$	Yes$^{(1)}$	Yes$^{(2)}$	Yes$^{(1)}$	Yes$^{(1)}$
Type 0	Yes$^{(2)}$	Yes$^{(1)}$	No$^{(3)}$	Yes$^{(1)}$	Yes$^{(1)}$

Proofs?

(1) proof via grammars, similar to context-free cases

(2) without proof

(3) proof in later chapters (part C)

Back to Chapter B13: Decidability

	Word problem	Emptiness problem	Equivalence problem	Intersection problem
Type 3	Yes	Yes	Yes	Yes
Type 2	Yes	Yes	No	No
Type 1	Yes$^{(1)}$	No$^{(3)}$	No$^{(2)}$	No$^{(2)}$
Type 0	No$^{(4)}$	No$^{(4)}$	No$^{(4)}$	No$^{(4)}$

Proofs?

(1) same argument we used for context-free languages

(2) because already undecidable for context-free languages

(3) without proof

(4) proofs in later chapters (part C)

Answers to Old Questions

Closure properties:

- H is Turing-recognizable (and thus type 0) but not decidable.

$\rightsquigarrow$ $\bar{H}$ is not Turing-recognizable, thus not type 0.

$\rightsquigarrow$ Type-0 languages are not closed under complement.

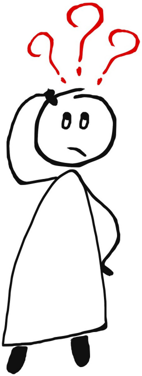
Decidability:

- H is type 0 but not decidable.

$\rightsquigarrow$ word problem for type-0 languages not decidable

$\rightsquigarrow$ emptiness, equivalence, intersection problem: later in exercises
(We are still missing some important results for this.)

Questions



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Summary

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- The halting problem is the language

$$H = \{w\#x \in \{0,1,\#\}^* \mid w, x \in \{0,1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

- The halting problem is Turing-recognizable but undecidable.
- The complement language $\bar{H}$ is an example of a language that is not even Turing-recognizable.