

Theory of Computer Science

B11. Turing Machines I

Gabriele Röger

University of Basel

April 3, 2024

Theory of Computer Science

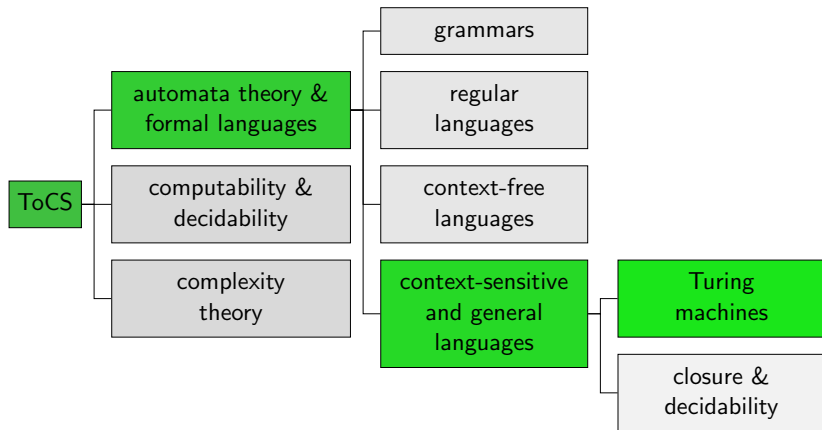
April 3, 2024 — B11. Turing Machines I

B11.1 Turing Machines

B11.2 Summary

B11.1 Turing Machines

Content of the Course



Automata for Type-1 and Type-0 Languages?



Finite automata recognize exactly the regular languages, push-down automata exactly the context-free languages. Are there automata models for context-sensitive and type-0 languages?

Yes! \rightsquigarrow Turing machines

Picture courtesy of [imagerymajestic](#) / [FreeDigitalPhotos.net](#)

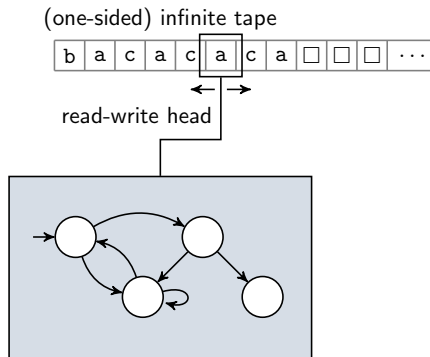
Alan Turing (1912–1954)



Picture courtesy of Jon Callas /
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- ▶ British logician, mathematician, cryptanalyst and computer scientist
- ▶ most important work (for us):
On Computable Numbers,
with an Application to the
Entscheidungsproblem
~> **Turing machines**
- ▶ collaboration on **Enigma decryption**
- ▶ conviction due to homosexuality;
pardoned by Elizabeth II in Dec. 2013
- ▶ **Turing award** most important
science award in computer science

Turing Machines: Conceptually



Turing Machine: Definition

Definition (Deterministic Turing Machine)

A (deterministic) **Turing machine (DTM)** is given by a 7-tuple $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$, where Q, Σ, Γ are all finite sets and

- ▶ Q is the set of **states**,
- ▶ Σ is the **input alphabet**, not containing the **blank symbol** \square ,
- ▶ Γ is the **tape alphabet**, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$,
- ▶ $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is the **transition function**,
- ▶ $q_0 \in Q$ is the **start state**,
- ▶ $q_{\text{accept}} \in Q$ is the **accept state**,
- ▶ $q_{\text{reject}} \in Q$ is the **reject state**, where $q_{\text{accept}} \neq q_{\text{reject}}$.

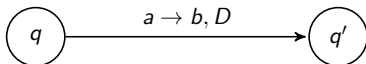
Turing Machine: Transition Function

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ be a DTM.

What is the Intuitive Meaning of the Transition Function δ ?

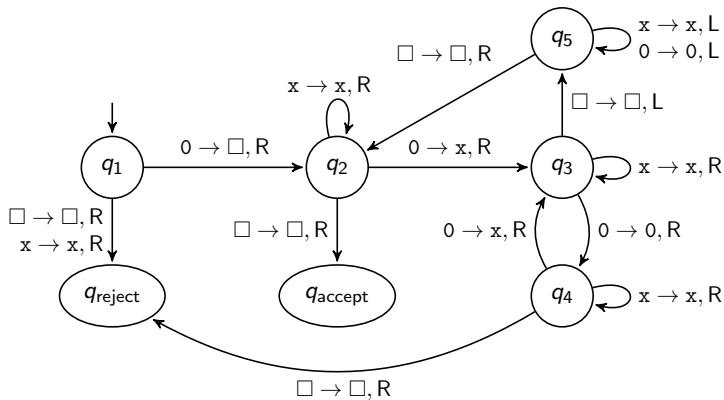
$\delta(q, a) = \langle q', b, D \rangle$:

- ▶ If M is in state q and reads a , then
- ▶ M transitions to state q' in the next step,
- ▶ replacing a with b ,
- ▶ and moving the head in direction $D \in \{L, R\}$, where:
 - ▶ **R**: one step to the **right**,
 - ▶ **L**: one step to the **left**, except if the head is on the left-most cell of the tape in which case there is no movement



Deterministic Turing Machine: Example

$\langle \{q_1, \dots, q_5, q_{\text{accept}}, q_{\text{reject}}\}, \{0\}, \{0, x, \square\}, \delta, q_1, q_{\text{accept}}, q_{\text{reject}} \rangle$



Turing Machine: Configuration

Definition (Configuration of a Turing Machine)

A **configuration** of a Turing machine

$$M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$$

is given by a triple $c \in \Gamma^* \times Q \times \Gamma^+$.

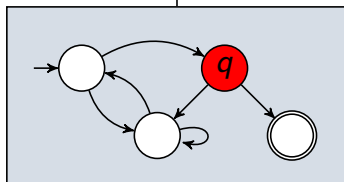
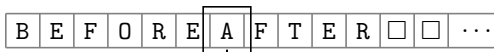
Configuration $\langle w_1, q, w_2 \rangle$ intuitively means that

- ▶ the non-empty or already visited part of the tape contains the word $w_1 w_2$,
- ▶ the read-write head is on the first symbol of w_2 , and
- ▶ the TM is in state q .

Turing Machine Configurations: Example

Example

configuration $\langle \text{BEFORE}, q, \text{AFTER}\square\square \rangle$.



Turing Machine Configurations: Start Configuration

Initially

- ▶ the TM is in start state q_0 ,
- ▶ the head is on the first tape cell, and
- ▶ the tape contains the input word w followed by an infinite number of \square entries.

The corresponding start configuration is $\langle \varepsilon, q_0, w \rangle$ if $w \neq \varepsilon$ and $\langle \varepsilon, q_0, \square \rangle$ if $w = \varepsilon$.

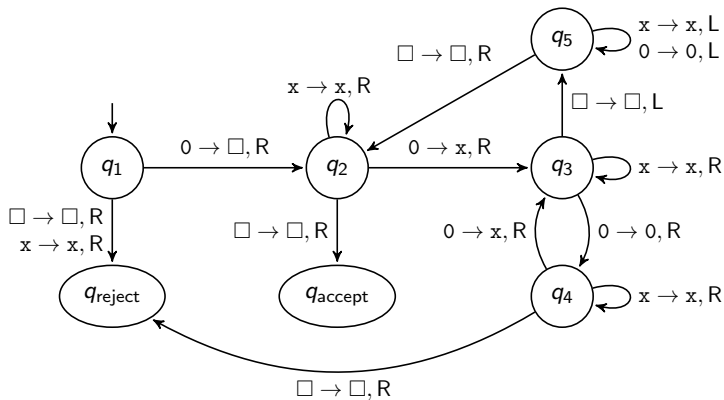
Turing Machine: Step

Definition (Transition/Step of a Turing Machine)

A DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ transitions from configuration c to configuration c' in one step ($c \vdash_M c'$) according to the following rules:

- ▶ $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_m c, q', b_2 \dots b_n \rangle$
if $\delta(q, b_1) = \langle q', c, R \rangle$, $m \geq 0$, $n \geq 2$
- ▶ $\langle a_1 \dots a_m, q, b_1 \rangle \vdash_M \langle a_1 \dots a_m c, q', \square \rangle$
if $\delta(q, b_1) = \langle q', c, R \rangle$, $m \geq 0$
- ▶ $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_{m-1}, q', a_m c b_2 \dots b_n \rangle$
if $\delta(q, b_1) = \langle q', c, L \rangle$, $m \geq 1$, $n \geq 1$
- ▶ $\langle \varepsilon, q, b_1 \dots b_n \rangle \vdash_M \langle \varepsilon, q', c b_2 \dots b_n \rangle$
if $\delta(q, b_1) = \langle q', c, L \rangle$, $n \geq 1$

Step: Exercise (Slido)



$\langle \square x, q_3, 00 \rangle \vdash ?$



DTM: Accepted Words

Intuitively, a DTM **accepts a word** if its computation terminates in the **accept state**.

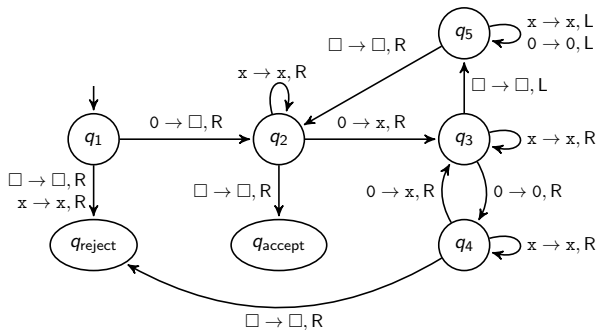
Definition (Words Accepted by a DTM)

DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ **accepts the word** $w = a_1 \dots a_n$ if there is a sequence of configurations c_0, \dots, c_k with

- 1 c_0 is the start configuration of M on input w ,
- 2 $c_i \vdash_M c_{i+1}$ for all $i \in \{0, \dots, k-1\}$, and
- 3 c_k is an accepting configuration,
i. e., a configuration with state q_{accept} .

Accepted Word: Example (Slido)

Does this Turing machine accept input 0000?



DTM: Recognized Language

Definition (Language Recognized by a DTM)

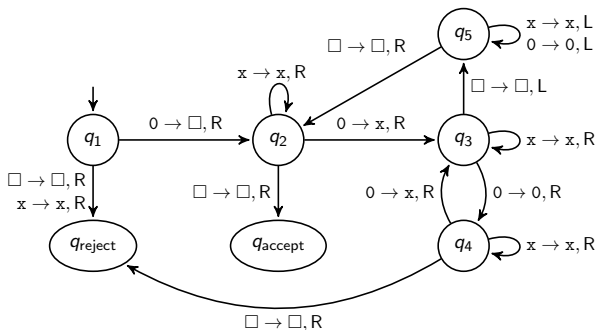
Let M be a deterministic Turing Machine

The **language recognized by M** (or **the language of M**) is defined as $\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

Definition (Turing-recognizable Language)

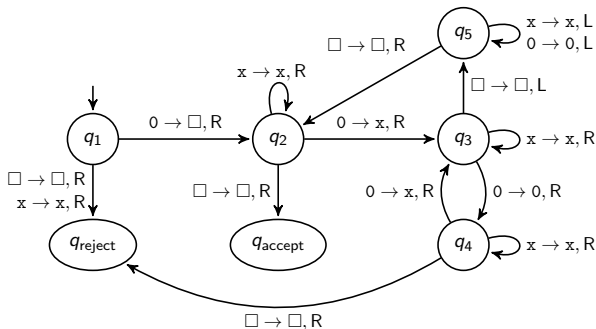
We call a language **Turing-recognizable** if some deterministic Turing machine recognizes it.

Turing Machine: Example



- 1 Sweep left to right across the tape, crossing off every other 0.
- 2 If in stage 1 the tape contained a single 0, accept.
- 3 If in stage 1 the tape contained more than one 0 and the number of 0s was odd, reject.
- 4 Return the head to the left end of the tape and go to stage 1.

Recognized Language: Example



What language does the Turing machine recognize?



Deciders

- ▶ A Turing machine either fails to accept an input
 - ▶ because it **rejects** it (entering q_{reject}) or
 - ▶ because it **loops** (= does not halt).
- ▶ A Turing machine that halts on all inputs (entering q_{reject} or q_{accept}) is called a **decider**.
- ▶ A decider that recognizes some language also is said to **decide** the language.

Definition (Turing-decidable Language)

We call a language **Turing-decidable** (or **decidable**) if some deterministic Turing machine decides it.

Exercise (if time)

Specify the state diagram of a DTM that decides language

$$L = \{w\#w \mid w \in \{0, 1\}^*\}.$$



Feel free to solve this together with your neighbour.

B11.2 Summary

Summary

- ▶ Turing machines only have finitely many states but an **unbounded tape** as “memory”.
- ▶ Alan Turing proposed them as a mathematical model for arbitrary algorithmic computations.
- ▶ In this role, we will revisit them in the parts on computability and complexity theory.