Theory of Computer Science

A2. Mathematical Foundations

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A2.1 Sets, Tuples, Relations

Sets

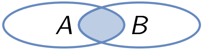
- set: unordered collection of distinguishable objects; each object contained at most once
- notations:
  - explicit, listing all elements, e. g. $A = \{1, 2, 3\}$
  - implicit, specifying a property characterizing all elements, e. g. $A = \{x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 3\}$
  - implicit, as a sequence with dots, e. g. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- $e \in M$: e is in set M (an element of the set)
- $e \notin M$: e is not in set M
- empty set $\emptyset = \{\}$
- cardinality $|M|$ of a finite set M: number of elements in M

Sets

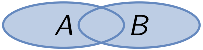
- $A \subseteq B$: A is a subset of B,
  i. e., every element of A is an element of B
- $A \subset B$: A is a strict subset of B,
  i. e., $A \subseteq B$ and $A \neq B$.
- power set $\mathcal{P}(M)$: set of all subsets of M
  e. g., $\mathcal{P}(\{a, b\}) =$
- Cardinality of power set of finite set S: $|\mathcal{P}(S)| =$

Set Operations

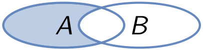
- intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



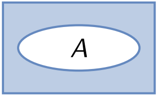
- union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



- difference $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



- complement $\overline{A} = B \setminus A$, where $A \subseteq B$ and B is the set of all considered objects (in a given context)



Tuples

- *k*-tuple: ordered sequence of k objects
- written $(o_1, \dots, o_k)$ or $\langle o_1, \dots, o_k \rangle$
- unlike sets, order matters ($\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$)
- objects may occur multiple times in a tuple
- objects contained in tuples are called components
- terminology:
  - $k = 2$: (ordered) pair
  - $k = 3$: triple
- if k is clear from context (or does not matter), often just called tuple

Cartesian Product

- for sets $M_1, M_2, \dots, M_n$, the Cartesian product $M_1 \times \dots \times M_n$ is the set $M_1 \times \dots \times M_n = \{\langle o_1, \dots, o_n \rangle \mid o_1 \in M_1, \dots, o_n \in M_n\}$.
- Example: $M_1 = \{a, b, c\}, M_2 = \{1, 2\}$, $M_1 \times M_2 = \{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \langle c, 1\rangle, \langle c, 2\rangle\}$
- special case: $M^k = M \times \dots \times M$ (k times)
- example with $M = \{1, 2\}$: $M^2 = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$

Relations

- an n-ary relation R over the sets $M_1, \dots, M_n$ is a subset of their Cartesian product: $R \subseteq M_1 \times \dots \times M_n$.
- example with $M = \{1, 2\}$:
  $R_{\leq} \subseteq M^2$ as $R_{\leq} = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$

A2.2 Functions

Functions

Definition (Total Function)

A (total) function $f : D \to C$ (with sets D, C)
maps every value of its domain D
to exactly one value of its codomain C.

Example

- $square : \mathbb{Z} \to \mathbb{Z}$ with $square(x) = x^2$
- $add : \mathbb{N}_0^2 \to \mathbb{N}_0$ with $add(x, y) = x + y$
- $add_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}$ with $add_{\mathbb{R}}(x, y) = x + y$

Functions: Example

Example

Let $Q = \{q_0, q_1, q_2, q_{\text{accept}}, q_{\text{reject}}\}$ and $\Gamma = \{0, 1, \square\}$.

Define $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \to Q \times \Gamma \times \{\text{L}, \text{R}\}$ by

δ	0	1	$\square$
q_0	$\langle q_0, 0, \text{R}\rangle$	$\langle q_0, 1, \text{R}\rangle$	$\langle q_1, \square, \text{L}\rangle$
q_1	$\langle q_2, 1, \text{L}\rangle$	$\langle q_1, 0, \text{L}\rangle$	$\langle q_{\text{reject}}, 1, \text{L}\rangle$
q_2	$\langle q_2, 0, \text{L}\rangle$	$\langle q_2, 1, \text{L}\rangle$	$\langle q_{\text{accept}}, \square, \text{R}\rangle$

Then, e. g., $\delta(q_0, 1) = \langle q_0, 1, \text{R}\rangle$

Partial Functions

Definition (Partial Function)

A partial function $f : X \rightarrow_{\mathrm{p}} Y$ maps every value in X to at most one value in Y.

If f does not map $x \in X$ to any value in Y, then f is undefined for x.

Example

$f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow_{\mathrm{p}} \mathbb{N}_0$ with

$$f(x, y) = \begin{cases} x - y & \text{if } y \leq x \\ \text{undefined} & \text{otherwise} \end{cases}$$

A2.3 Summary

Summary

- sets: unordered, contain every element at most once
- tuples: ordered, can contain the same object multiple times
- Cartesian product: $M_1 \times \cdots \times M_n$ set of all n-tuples where the i-th component is in M_i
- function $f : X \to Y$ maps every value in X to exactly one value in Y
- partial function $g : X \to_{\mathrm{p}} Y$ may be undefined for some values in X