

# Foundations of Artificial Intelligence

## E1. Propositional Logic: Syntax and Semantics

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April 17, 2024

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E1.1 Motivation

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# Propositional Logic: Overview

## Chapter overview: propositional logic

- ▶ E1. Syntax and Semantics
- ▶ E2. Equivalence and Normal Forms
- ▶ E3. Reasoning and Resolution
- ▶ E4. DPLL Algorithm
- ▶ E5. Local Search and Outlook

# Classification

classification:

Propositional Logic

environment:

- ▶ **static** vs. dynamic
- ▶ **deterministic** vs. non-deterministic vs. stochastic
- ▶ **fully** vs. partially vs. not **observable**
- ▶ **discrete** vs. continuous
- ▶ **single-agent** vs. multi-agent

problem solving method:

- ▶ problem-specific vs. **general** vs. learning

(applications also in more complex environments)

# E1.1 Motivation

# Propositional Logic: Motivation

## propositional logic

- ▶ modeling and representing problems and knowledge
- ▶ basis for **general** problem descriptions and solving strategies  
     $\rightsquigarrow$  automated planning (Part F)
- ▶ allows for automated **reasoning**

**German:** Aussagenlogik, automatisches Schliessen

# Relationship to CSPs

- ▶ **previous part:** constraint satisfaction problems
- ▶ satisfiability problem in propositional logic can be viewed as **non-binary CSP** over  $\{\mathbf{F}, \mathbf{T}\}$
- ▶ formula encodes constraints
- ▶ solution: satisfying assignment of values to variables
- ▶ backtracking with inference  $\approx$  DPLL ([Chapter E4](#))

# Propositional Logic: Description of State Spaces

propositional variables for missionaries and cannibals problem:

two-missionaries-are-on-left-shore

one-cannibal-is-on-left-shore

boat-is-on-left-shore

...

- ▶ problem description for general problem solvers
- ▶ states represented as truth values of atomic **propositions**

German: Aussagenvariablen



# Propositional Logic: Intuition

**propositions:** atomic statements over the world that cannot be divided further

Propositions with **logical connectives** like “and”, “or” and “not” form the propositional formulas.

**German:** logische Verknüpfungen

# Syntax and Semantics

Today, we define **syntax** and **semantics** of propositional logic.  
 $\rightsquigarrow$  **reminder** from Discrete Mathematics in Computer Science

**syntax:**

- ▶ defines which **symbols** are allowed in formulas  
 $(, ), \neg, \wedge, A, B, C, X, \heartsuit, \rightarrow, \nearrow, \dots?$
- ▶ ... and which **sequences** of these symbols are correct formulas  
 $(A \wedge B), ((A) \wedge B), \wedge)A(B, \dots?$

**semantics:**

- ▶ defines the **meaning** of formulas
- ▶ uses **interpretations** to describe a possible world  
 $I = \{A \mapsto \mathbf{T}, B \mapsto \mathbf{F}\}$
- ▶ tells us which formulas are true in which worlds

## E1.2 Syntax

# Alphabet of Propositions

- ▶ Logical formulas use an **alphabet  $\Sigma$  of propositions**, for example  $\Sigma = \{P, Q, R, S\}$  or  $\Sigma = \{X_1, X_2, X_3, \dots\}$ .
- ▶ We do not mention the alphabet in the following.
- ▶ More formally, all definitions are parameterized by  $\Sigma$ .

German: Alphabet

# Syntax

## Definition (propositional formula)

- ▶  $\top$  and  $\perp$  are formulas (**constant true/constant false**).
- ▶ Every proposition in  $\Sigma$  is a formula (**atomic formula**).
- ▶ If  $\varphi$  is a formula, then  $\neg\varphi$  is a formula (**negation**).
- ▶ If  $\varphi$  and  $\psi$  are formulas, then so are
  - ▶  $(\varphi \wedge \psi)$  (**conjunction**)
  - ▶  $(\varphi \vee \psi)$  (**disjunction**)
  - ▶  $(\varphi \rightarrow \psi)$  (**implication**)

**German:** aussagenlogische Formel, konstant wahr/falsch,  
atomare Formel, Konjunktion, Disjunktion, Implikation

**Note:** minor differences to Discrete Mathematics course

# Abbreviating Notations: Omitting Parenthesis

may omit redundant parentheses:

- ▶ outer parentheses of formula:
  - ▶  $(P \wedge Q) \vee R$  instead of  $((P \wedge Q) \vee R)$
- ▶ multiple conjunctions/disjunctions:
  - ▶  $P \wedge Q \wedge \neg R \wedge S$  instead of  $((((P \wedge Q) \wedge \neg R) \wedge S))$
- ▶ implicit **binding strength**:  $(\neg) > (\wedge) > (\vee) > (\rightarrow)$ :
  - ▶  $P \vee Q \wedge R$  instead of  $P \vee (Q \wedge R)$
  - ▶ use responsibly

# Abbreviating Notations: Prefix Notation

prefix notations used like  $\sum$  for sums:

▶  $\bigvee_{i=1}^4 X_i$  instead of  $(X_1 \vee X_2 \vee X_3 \vee X_4)$

▶  $\bigwedge_{i=1}^3 Y_i$  instead of  $(Y_1 \wedge Y_2 \wedge Y_3)$

## E1.3 Semantics



# Intuition for Semantics

A formula is **true** or **false** depending on the **interpretation** of the propositions.

## Semantics: Intuition

- ▶ A proposition  $P$  is either true or false.  
The truth value of  $P$  is determined by an **interpretation**.
- ▶ The truth value of a formula follows from the truth values of the propositions.

## Example

example interpretations for  $\varphi = (P \vee Q) \wedge R$ :

- ▶ If  $P$  and  $Q$  are false and  $R$  is true, then  $\varphi$  is false.
- ▶ If  $P$  is false and  $Q$  and  $R$  are true, then  $\varphi$  is true.

# Interpretations

## Definition (interpretation)

An **interpretation**  $I$  is a function  $I : \Sigma \rightarrow \{\mathbf{T}, \mathbf{F}\}$ .

Interpretations are sometimes called **truth assignments**.

**German:** Interpretation/Belegung/Wahrheitsbelegung

# The Semantics of Formulas

When is a formula  $\varphi$  true under interpretation  $I$ ?  
 symbolically: When does  $I \models \varphi$  hold?

## Definition (Models and the $\models$ Relation)

The relation “ $\models$ ” is a relation between interpretations and formulas and is defined as follows:

- ▶  $I \models \top$  and  $I \not\models \perp$
- ▶  $I \models P$  if  $I(P) = \mathbf{T}$  for  $P \in \Sigma$
- ▶  $I \models \neg\varphi$  if  $I \not\models \varphi$
- ▶  $I \models (\varphi \wedge \psi)$  if  $I \models \varphi$  and  $I \models \psi$
- ▶  $I \models (\varphi \vee \psi)$  if  $I \models \varphi$  or  $I \models \psi$
- ▶  $I \models (\varphi \rightarrow \psi)$  if  $I \not\models \varphi$  or  $I \models \psi$

If  $I \models \varphi$  ( $I \not\models \varphi$ ), we say  $\varphi$  is **true (false)** under  $I$ .

# Examples

## Example (Interpretation $I$ )

$$I = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{F}\}$$

## Which formulas are true under $I$ ?

- ▶  $\varphi_1 = \neg(P \wedge Q) \wedge (R \wedge \neg S)$ . Does  $I \models \varphi_1$  hold?
- ▶  $\varphi_2 = (P \wedge Q) \wedge \neg(R \wedge \neg S)$ . Does  $I \models \varphi_2$  hold?
- ▶  $\varphi_3 = (R \rightarrow P)$ . Does  $I \models \varphi_3$  hold?

## E1.4 Properties of Formulas

# Models of Formulas and Sets of Formulas

## Definition (model)

An interpretation  $I$  is called a **model** of  $\varphi$  if  $I \models \varphi$ .

## Definition ( $I \models \Phi$ )

Let  $\Phi$  be a set of propositional formulas.

We write  $I \models \Phi$  if  $I \models \varphi$  for all  $\varphi \in \Phi$ .

Such an interpretation  $I$  is called a **model** of  $\Phi$ .

If  $I$  is a model of formula  $\varphi$ , we also say “ $I$  satisfies  $\varphi$ ” or “ $\varphi$  holds under  $I$ ” (similarly for sets of formulas  $\Phi$ ).

**German:** Modell, erfüllt, gilt unter

# Satisfiable, Unsatisfiable, Falsifiable, Valid

## Definition (satisfiable etc.)

A formula  $\varphi$  is called

- ▶ **satisfiable** if there exists a model for  $\varphi$
- ▶ **unsatisfiable** if there exists no model for  $\varphi$
- ▶ **valid** (= a **tautology**) if all interpretations are models of  $\varphi$
- ▶ **falsifiable** if not all interpretations are models of  $\varphi$

**German:** erfüllbar, unerfüllbar, allgemeingültig (gültig, Tautologie),  
falsifizierbar

# Truth Tables

## Truth Tables

How to determine automatically if a given formula is (un)satisfiable, valid, falsifiable?

$\rightsquigarrow$  simple method: **truth tables**

**example:** Is  $\varphi = ((P \vee H) \wedge \neg H) \rightarrow P$  valid?

| $P$      | $H$      | $P \vee H$ | $((P \vee H) \wedge \neg H)$ | $((P \vee H) \wedge \neg H) \rightarrow P$ |
|----------|----------|------------|------------------------------|--|
| <b>F</b> | <b>F</b> | <b>F</b>   | <b>F</b>                     | <b>T</b>                                   |
| <b>F</b> | <b>T</b> | <b>T</b>   | <b>F</b>                     | <b>T</b>                                   |
| <b>T</b> | <b>F</b> | <b>T</b>   | <b>T</b>                     | <b>T</b>                                   |
| <b>T</b> | <b>T</b> | <b>T</b>   | <b>F</b>                     | <b>T</b>                                   |

$I \models \varphi$  for all interpretations  $I$ , hence  $\varphi$  is valid.

► Is it satisfiable/unsatisfiable/falsifiable?



# Terminology (Side Note)

What does “ $\varphi$  is true” mean?

- ▶ not formally defined
- ▶ the statement misses an interpretation
  - ▶ could be meant as “in the obvious interpretation”  
in some cases
  - ▶ or as “in all possible interpretations” (tautology)
- ▶ imprecise language  $\rightsquigarrow$  avoid

# E1.5 Summary

# Summary

- ▶ **Propositional logic** forms the basis for a general representation of problems and knowledge.
- ▶ **Propositions** (atomic formulas) are statements over the world that cannot be divided further.
- ▶ **Propositional formulas** combine constant and atomic formulas with  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\rightarrow$  to more complex statements.
- ▶ **Interpretations** determine which atomic formulas are true and which ones are false.
- ▶ Interpretations making a formula true are called **models**.
- ▶ important properties a formula may have:  
**satisfiable, unsatisfiable, valid, falsifiable**