

Foundations of Artificial Intelligence

D3. Constraint Satisfaction Problems: Backtracking

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Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- D1–D2. Introduction
- D3–D5. Basic Algorithms
 - D3. Backtracking
 - D4. Arc Consistency
 - D5. Path Consistency
- D6–D7. Problem Structure

CSP Algorithms

CSP Algorithms

In the following chapters, we consider **algorithms for solving** constraint networks.

basic concepts:

- **search**: check partial assignments systematically
- **backtracking**: discard inconsistent partial assignments
- **inference**: derive equivalent, but tighter constraints to reduce the size of the search space

Naive Backtracking

Naive Backtracking (= Without Inference)

```
function NaiveBacktracking( $\mathcal{C}, \alpha$ ):
```

```
   $\langle V, \text{dom}, (R_{uv}) \rangle := \mathcal{C}$ 
```

```
  if  $\alpha$  is inconsistent with  $\mathcal{C}$ :  
    return inconsistent
```

```
  if  $\alpha$  is a total assignment:  
    return  $\alpha$ 
```

```
  select some variable  $v$  for which  $\alpha$  is not defined
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```
  for each  $d \in \text{dom}(v)$  in some order:
```

```
     $\alpha' := \alpha \cup \{v \mapsto d\}$ 
```

```
     $\alpha'' := \text{NaiveBacktracking}(\mathcal{C}, \alpha')$ 
```

```
    if  $\alpha'' \neq \text{inconsistent}$ :  
      return  $\alpha''$ 
```

```
  return inconsistent
```

input: constraint network \mathcal{C} and partial assignment α for \mathcal{C}
(first invocation: empty assignment $\alpha = \emptyset$)

result: solution of \mathcal{C} or **inconsistent**

Running Example

Full Formal Model of Running Example

$\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$ with

- variables:

$$V = \{w, x, y, z\}$$

- domains:

$$\text{dom}(w) = \text{dom}(y) = \{1, 2, 3, 4\}$$

$$\text{dom}(x) = \text{dom}(z) = \{1, 2, 3\}$$

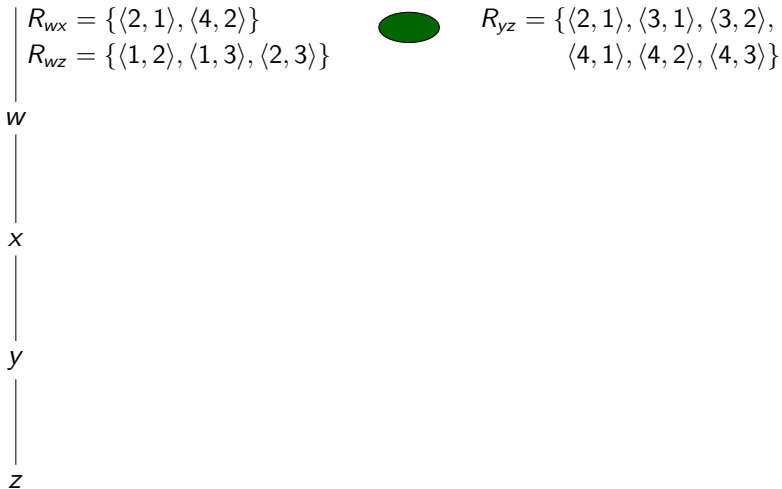
- constraints:

$$R_{wx} = \{\langle 2, 1 \rangle, \langle 4, 2 \rangle\}$$

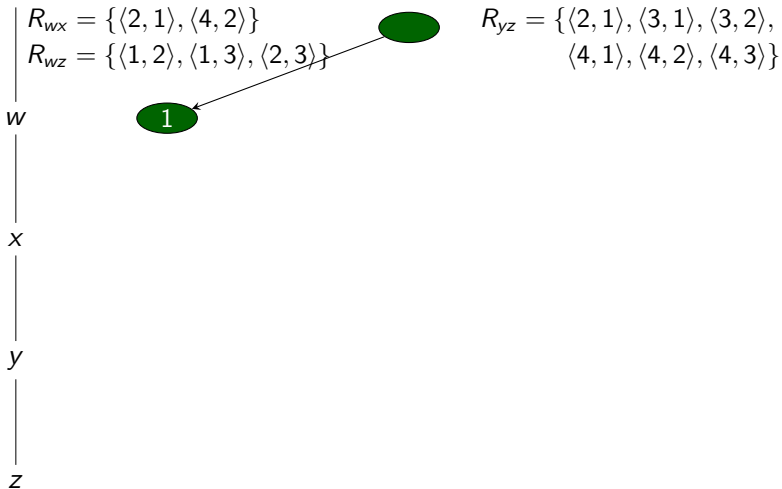
$$R_{wz} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}$$

$$R_{yz} = \{\langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \\ \langle 4, 1 \rangle, \langle 4, 2 \rangle, \langle 4, 3 \rangle\}$$

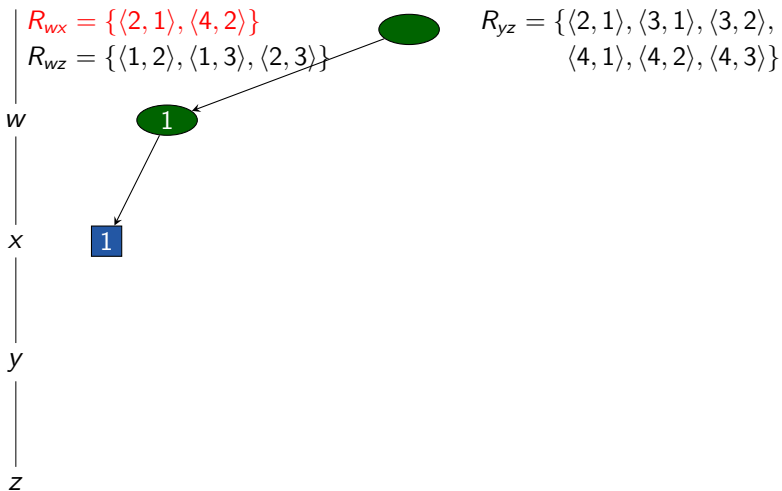
Running Example: Search Tree



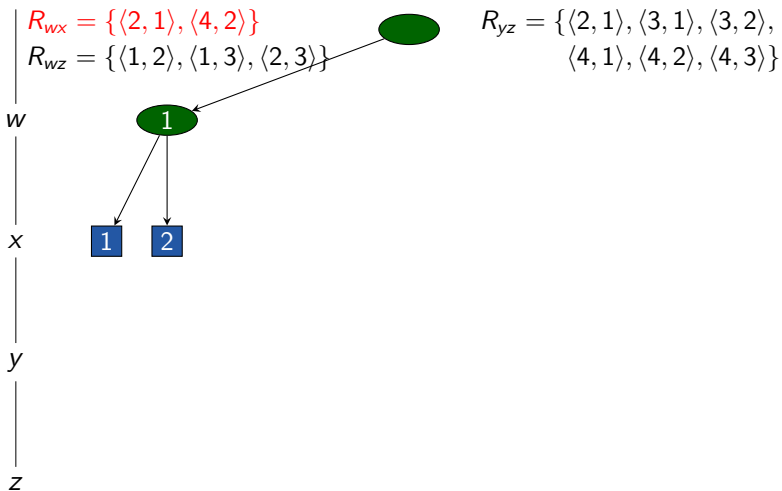
Running Example: Search Tree



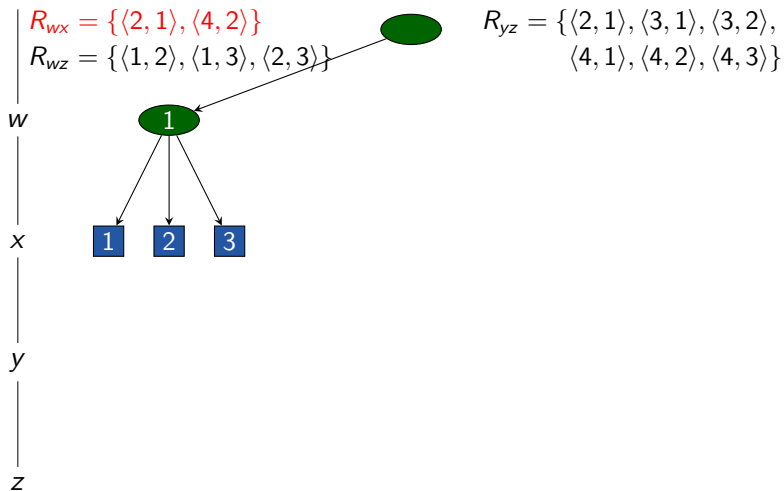
Running Example: Search Tree



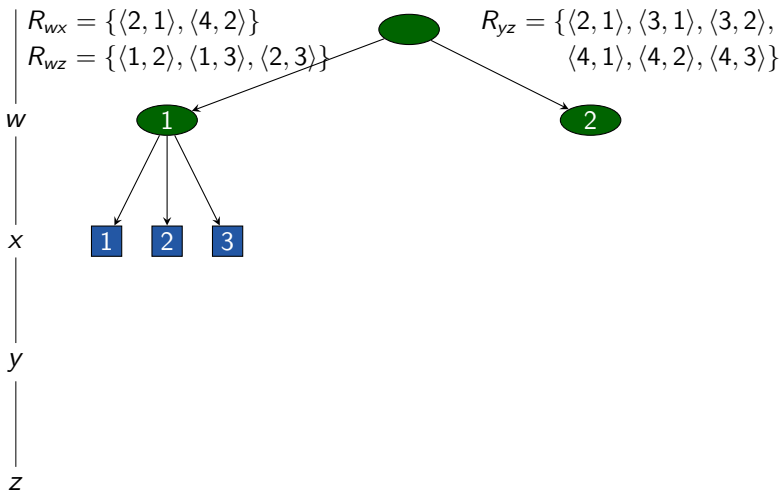
Running Example: Search Tree



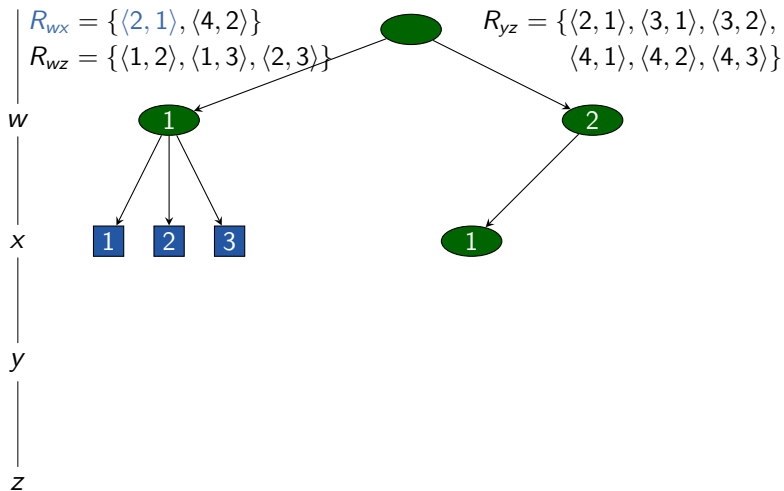
Running Example: Search Tree



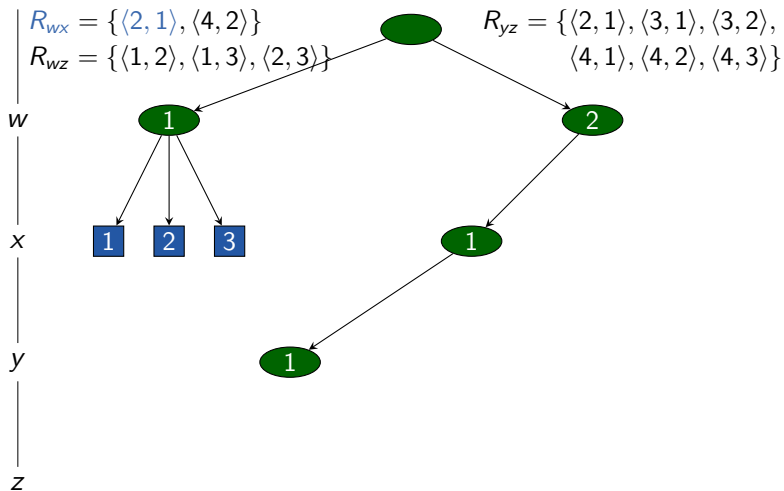
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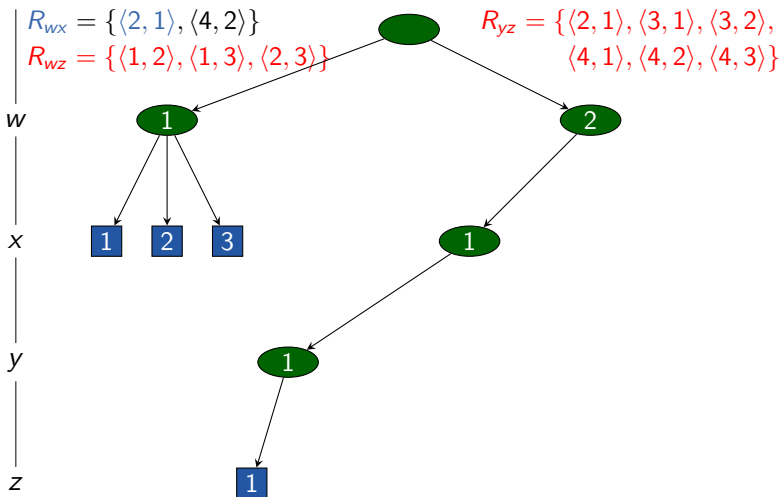
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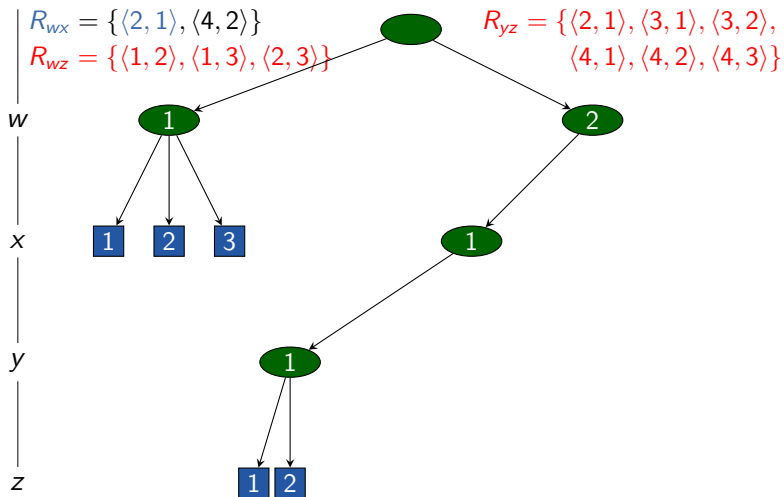
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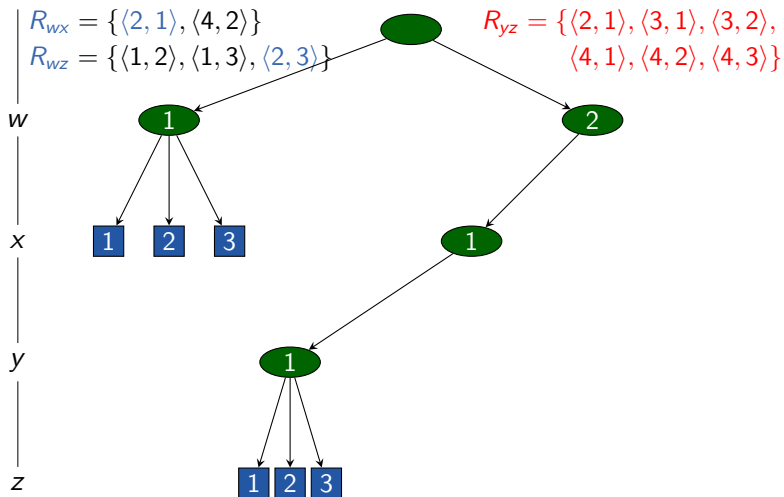
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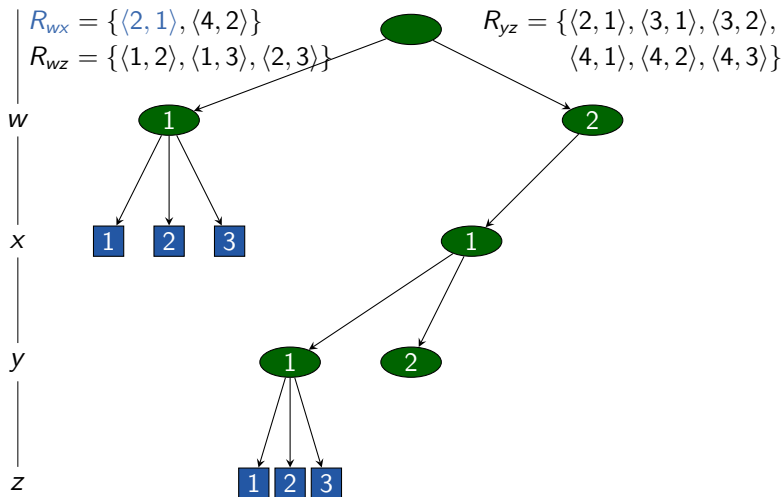
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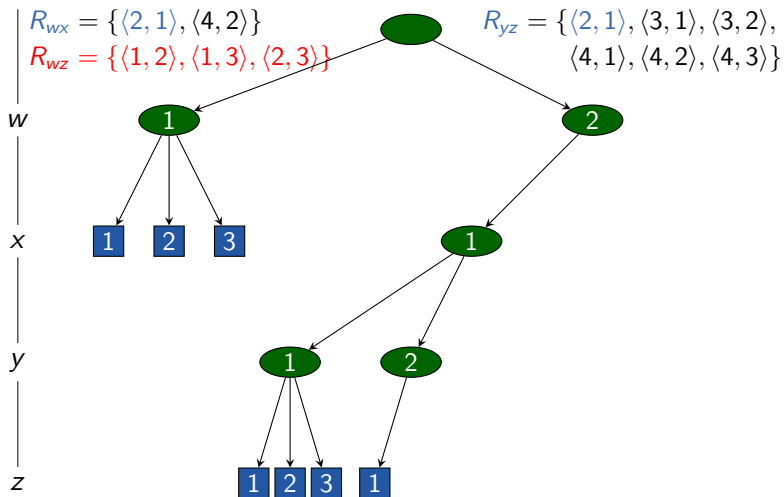
Running Example: Search Tree



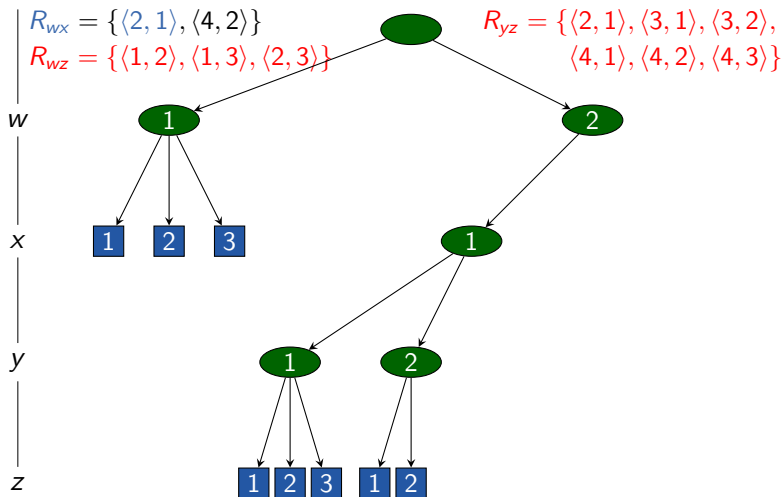
Running Example: Search Tree



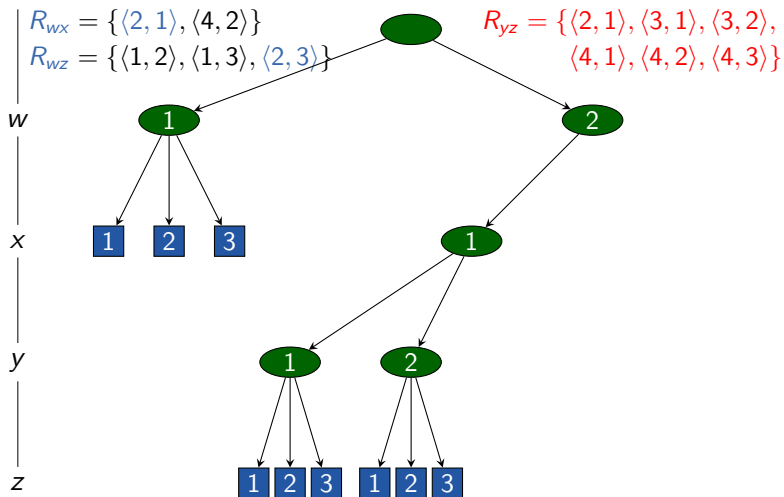
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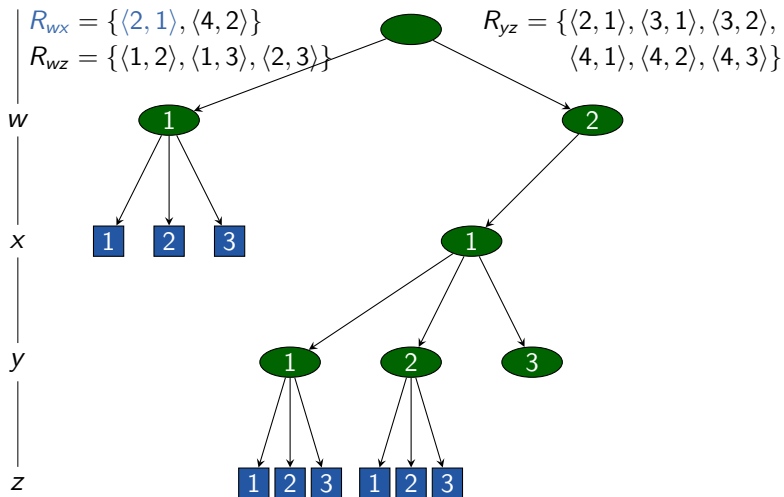
Running Example: Search Tree



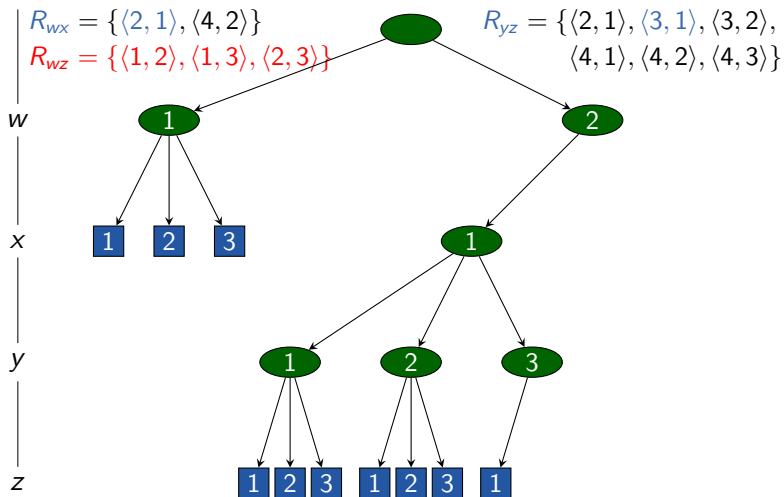
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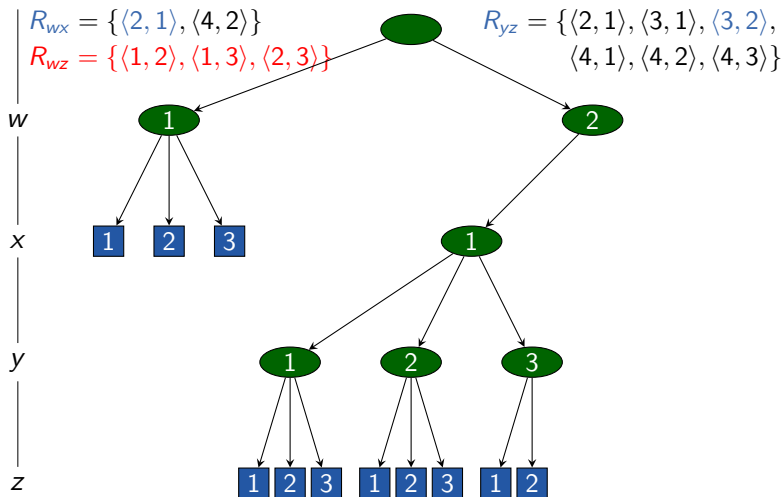
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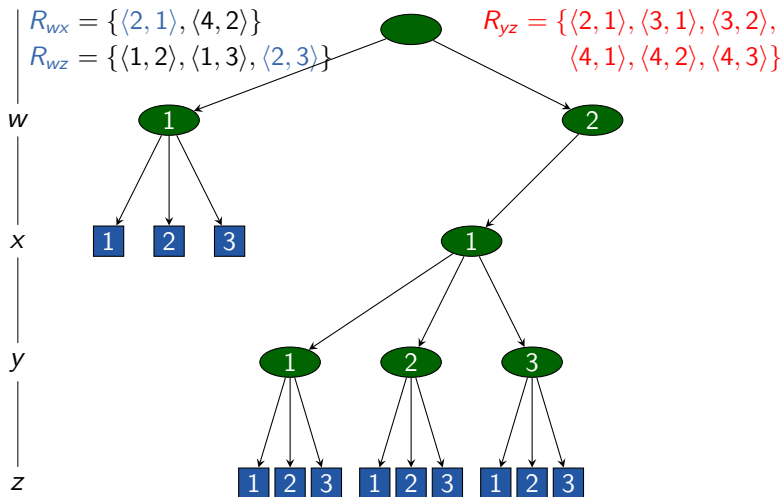
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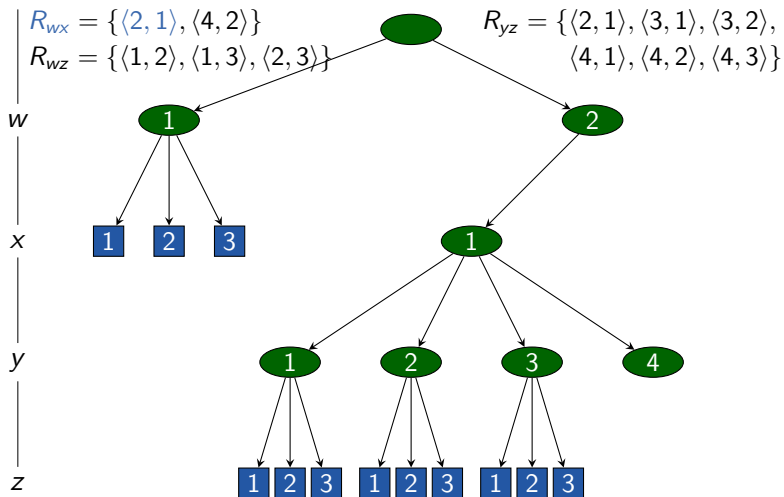
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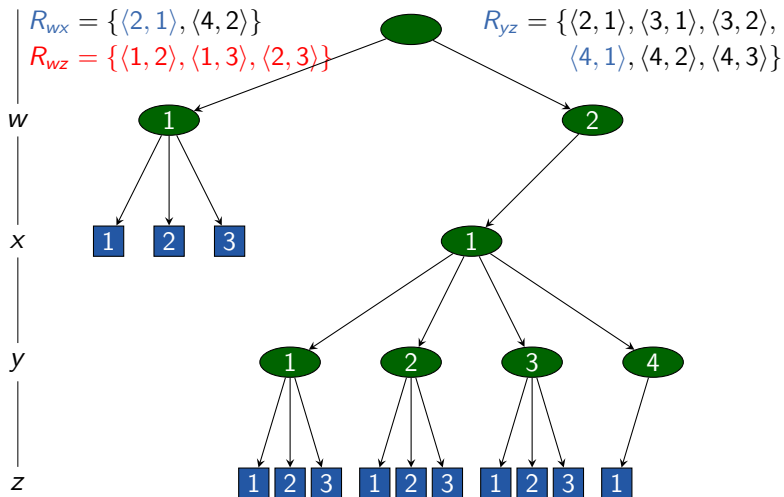
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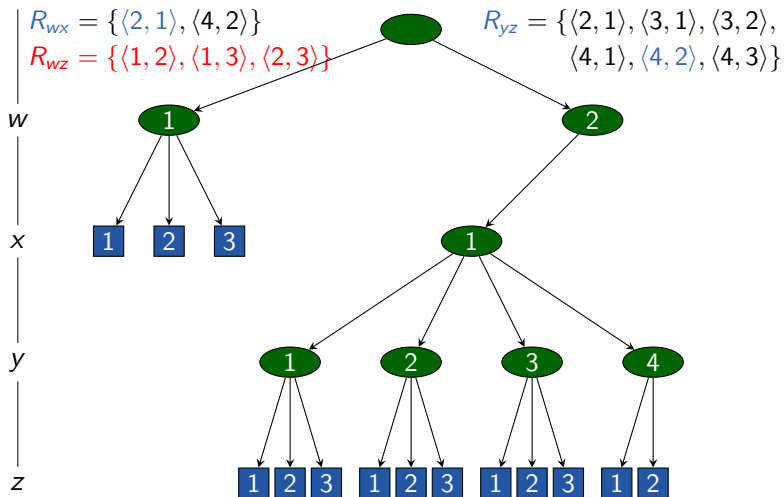
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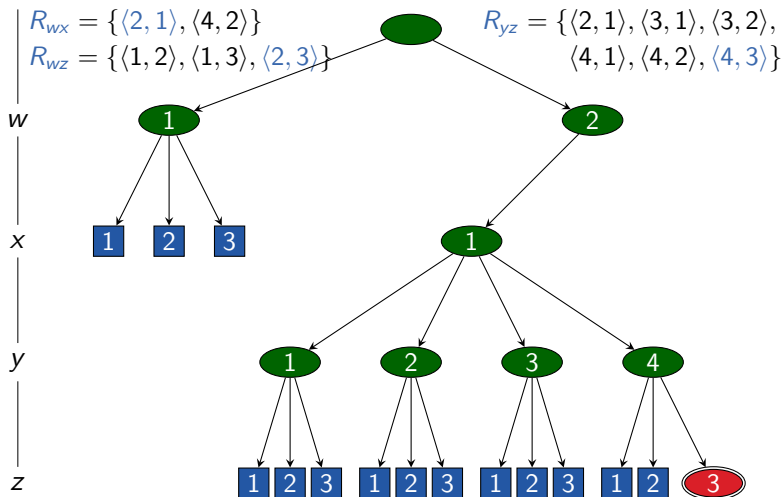
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Running Example: Search Tree



Is This a New Algorithm?

We have already seen this algorithm:

Backtracking corresponds to depth-first search (Chapter B8)

with the following state space:

- **states:** partial assignments
- **initial state:** empty assignment \emptyset
- **goal states:** consistent total assignments
- **actions:** $assign_{v,d}$ assigns value $d \in \text{dom}(v)$ to variable v
- **action costs:** all 0 (all solutions are of equal quality)
- **transitions:**
 - for each **non-total consistent** assignment α ,
choose variable $v = \text{select}(\alpha)$ that is unassigned in α
 - transition $\alpha \xrightarrow{assign_{v,d}} \alpha \cup \{v \mapsto d\}$ for each $d \in \text{dom}(v)$

Why Depth-First Search?

Depth-first search is particularly well-suited for CSPs:

- path length **bounded** (by the number of variables)
- solutions located at **the same depth** (lowest search layer)
- state space is directed **tree**, initial state is the root
 \rightsquigarrow **no duplicates** (Why?)

Hence none of the problematic cases for depth-first search occurs.

Naive Backtracking: Discussion

- Naive backtracking often has to exhaustively explore **similar** search paths (i.e., partial assignments that are identical except for a few variables).
 - “Critical” variables are not recognized and hence considered for assignment (too) late.
 - Decisions that necessarily lead to constraint violations are only recognized when all variables involved in the constraint have been assigned.
- ↪ more intelligence by **focusing on critical decisions** and by **inference** of consequences of previous decisions

Variable and Value Orders

Naive Backtracking

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Variable Orders

- Backtracking does not specify in which order **variables** are considered for assignment.
- Such orders can strongly influence the search space size and hence the search performance.
 ↪ **example:** exercises
- Eventually we have to assign all variables
 ↪ prefer critical assignments (**fail early**)

German: Variablenordnung

Value Orders

- Backtracking does not specify in which order the **values** of the selected variable v are considered.
- This is not as important because it **does not matter** in subtrees without a solution. (Why not?)
- **If** there is a solution in the subtree, then ideally a value that leads to a solution should be chosen.
 \rightsquigarrow prefer promising assignments

German: Werteordnung

Static vs. Dynamic Orders

we distinguish:

- **static** orders (fixed prior to search)
- **dynamic** orders (selected variable or value order depends on the search state)

comparison:

- dynamic orders obviously more powerful
- static orders \rightsquigarrow no computational overhead during search

The following ordering criteria can be used statically, but are more effective combined with inference (\rightsquigarrow later) and used dynamically.

Variable Orders

two common variable ordering criteria:

- **minimum remaining values:**
prefer variables that have small **domains**
 - **intuition:** few subtrees \rightsquigarrow smaller tree
 - **extreme case:** only **one** value \rightsquigarrow forced assignment
- **most constraining variable:**
prefer variables contained in **many** nontrivial constraints
 - **intuition:** constraints tested early
 \rightsquigarrow inconsistencies recognized early \rightsquigarrow smaller tree

combination: use minimum remaining values criterion,
then most constraining variable criterion to break ties

Value Orders

Definition (conflict)

Let $\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$ be a constraint network.
For variables $v \neq v'$ and values $d \in \text{dom}(v)$, $d' \in \text{dom}(v')$,
the assignment $v \mapsto d$ is **in conflict** with $v' \mapsto d'$ if $\langle d, d' \rangle \notin R_{vv'}$.

value ordering criterion for partial assignment α
and selected variable v :

- **minimum conflicts**: prefer values $d \in \text{dom}(v)$
such that $v \mapsto d$ causes as few conflicts as possible
with variables that are unassigned in α

Summary

Summary: Backtracking

basic search algorithm for constraint networks: **backtracking**

- extends the (initially empty) partial assignment step by step until an **inconsistency** or a **solution** is found
- is a form of **depth-first search**
- depth-first search particularly well-suited because state space is directed tree and all solutions at same (known) depth

Summary: Variable and Value Orders

- **Variable orders** influence the performance of backtracking significantly.
 - goal: **critical** decisions as early as possible
- **Value orders** influence the performance of backtracking on **solvable** constraint networks significantly.
 - goal: **most promising** assignments first