Algorithms and Data Structures

C7. Graphs: Outlook

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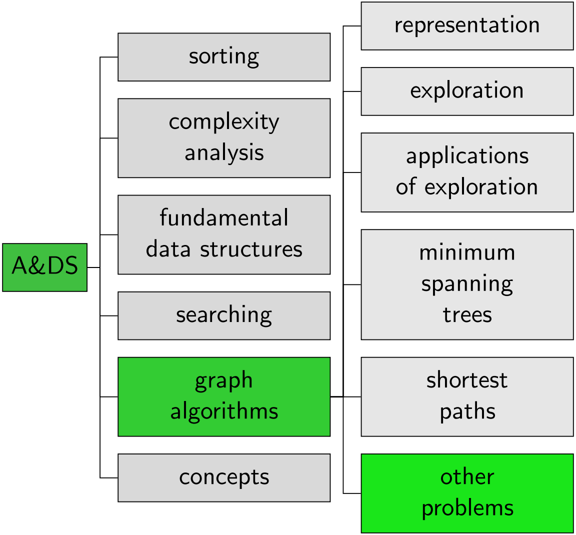
Other Graph Problems

Content of the Course



Crash Course Complexity Theory

- Decision problems: Seeking a Yes/No answer
  Given weighted graph, vertices s, t and number K.
  Is there a path from s to t that costs at most K?
- Search problem: Seeking an actual solution
  Given weighted graph and vertices s, t.
  Find a shortest path from s to t.

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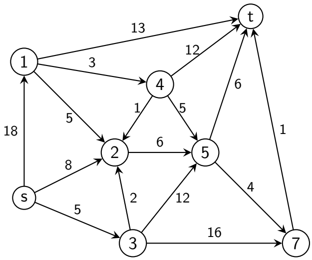
- NP-complete decision problems: NP-hard & in NP
- NP-equivalent search problems: corresponding decision problem NP-complete.

Flows in Graphs I

Definition (Flow Network)

A flow network $N = (G, s, t, k)$ is given by

- a directed graph $G = (V, E)$,
- a source $s \in V$,
- a sink $t \in V$, and
- a capacity function $k : E \to \mathbb{R}_+^\infty$.



Flows in Graphs II

Definition (Flow)

An s-t flow f assigns every edge a value from $\mathbb{R}_{\geq 0}$, where

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$$\sum_{\substack{(u,w)\in E \\ w=v}} f((u,w)) = \sum_{\substack{(u,w)\in E \\ u=v}} f((u,w)) \text{ for all } v \in V \setminus \{s,t\}$$

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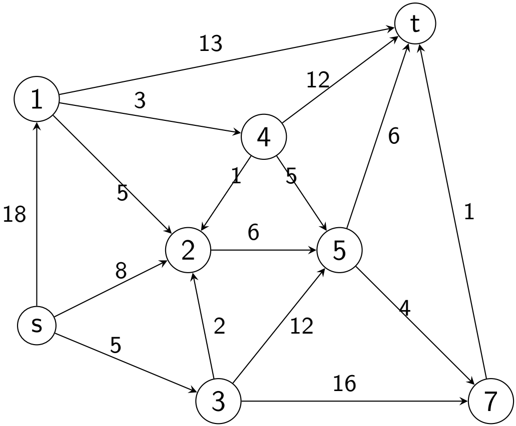
The value of the flow is the net flow into the sink:

$$|f| = \sum_{\substack{(u,w)\in E\\ w=t}} f((u,w)) - \sum_{\substack{(u,w)\in E\\ u=t}} f((u,w))$$

Example



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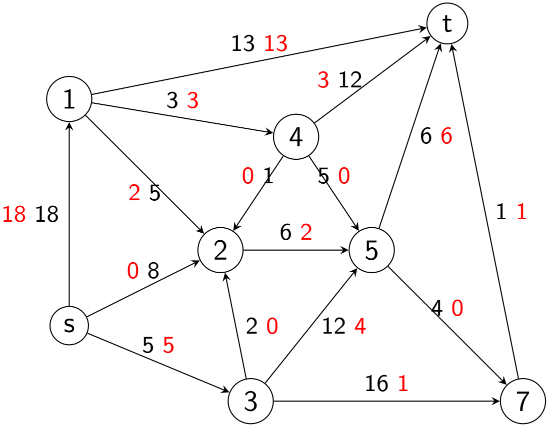

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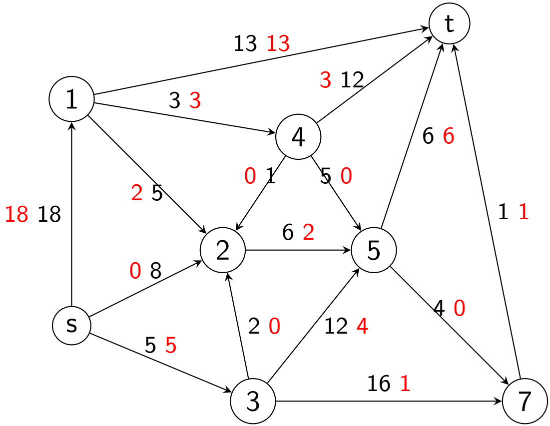

How hard is it to find a maximal flow?

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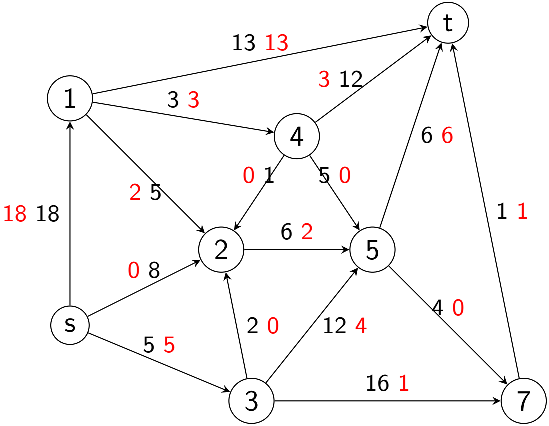


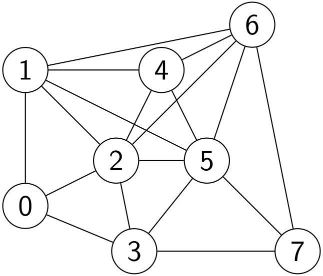
How hard is it to find a maximal flow?

For example with the Edmonds-Karp algorithm in $O(|E|^2|V|)$

Cliques

Definition (Clique)

A clique in an undirected graph (V, E) is a subset $C \subseteq V$ of the vertices such that each pair of distinct vertices in C is connected by an edge: for $u, v \in C$ with $u \neq v$ it holds that $\{u, v\} \in E$.



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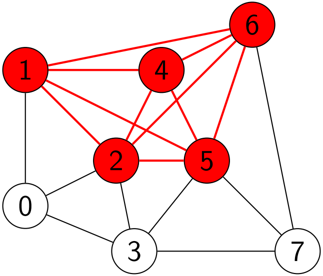

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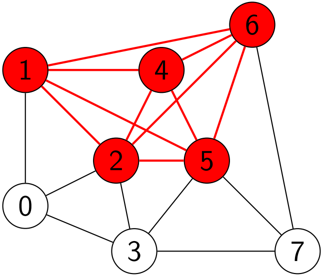

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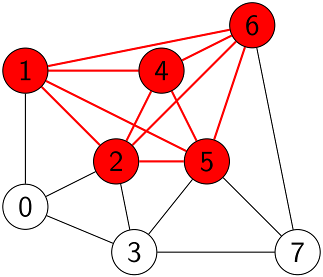

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NP-equivalent

Graph Isomorphism

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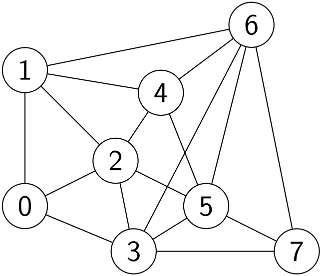
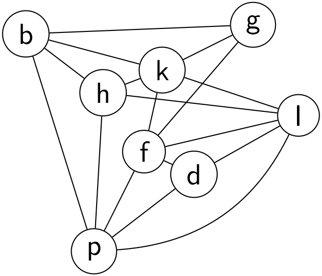
Two graphs are isomorphic, if they are identical up to renaming vertices.





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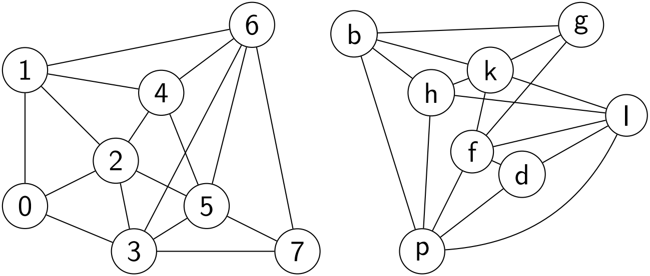

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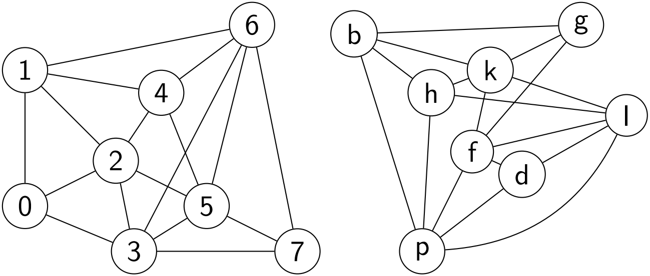


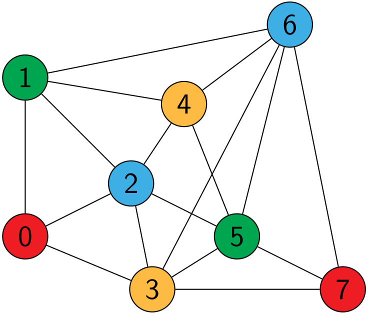
How hard is it to decide whether two given graphs are isomorphic?

In NP, but unknown whether in P and/or NP-complete

Graph Coloring

Definition (k-Colorability)

An undirected graph $G = (V, E)$ is k-colorable ($k \in \mathbb{N}$), if there is a coloring $f : V \to \{1, \dots, k\}$ with $f(v) \neq f(w)$ for all $\{v, w\} \in E$.



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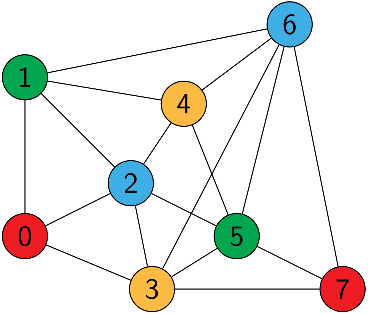

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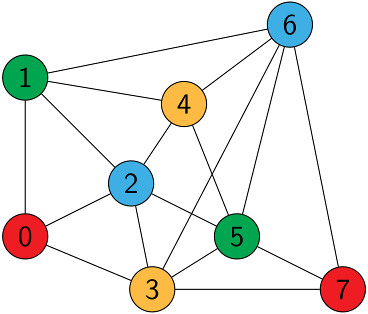

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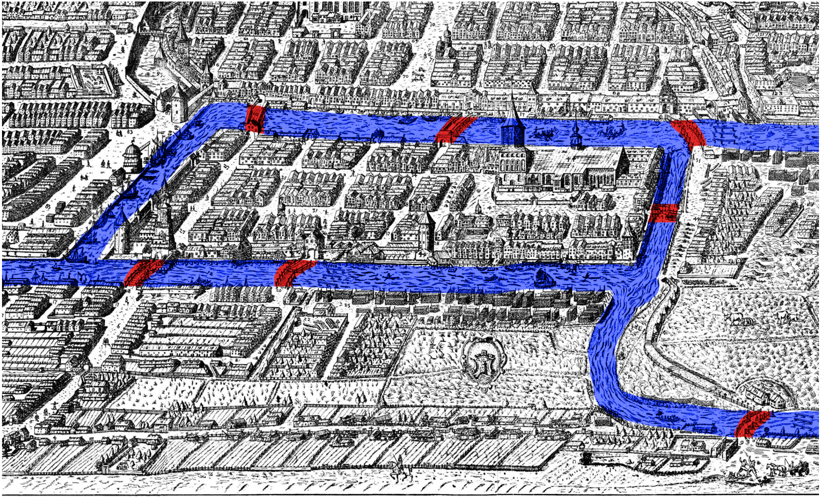
NP-complete

Seven Bridges of Königsberg

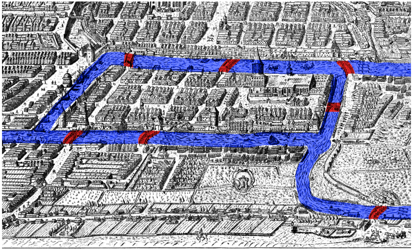


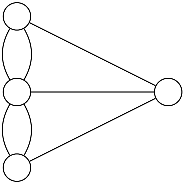
Is there a walk through the city crossing each bridge exactly once?

Eulerian Trail

Definition (Eulerian Trail)

An Eulerian trail is a path that uses every edge exactly once.





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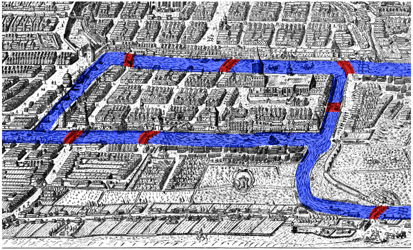

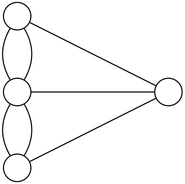

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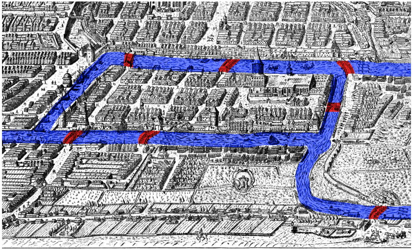

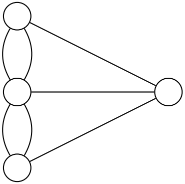

How hard is it to decide whether a graph has an Eulerian trail?

Has Eulerian trail iff exactly zero or two vertices have odd degree, and all of its vertices with nonzero degree belong to a single connected component.

Quiz

Quiz

kahoot.it