Algorithms and Data Structures

C5. Shortest Paths: Foundations

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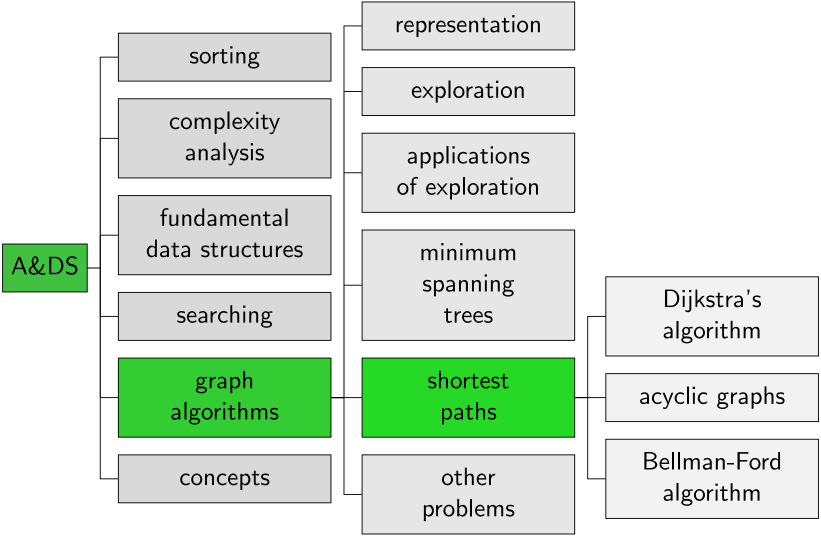
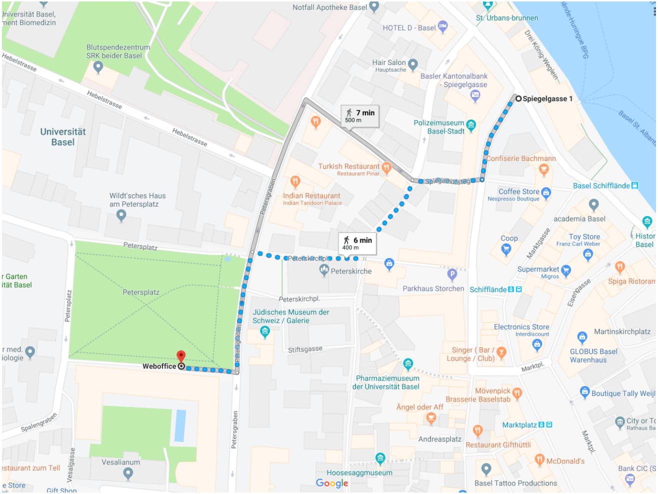
Content of the Course



Introduction

Google Maps


Notfall Apotheke Basel
St. Urbans-brunnen
HOTEL D - Basel
Blutspendezentrum
SRK beider Basel
Hebelstrasse
Hair Salon
Hauptsache
Basler Kantonalbank
- Spiegelgasse
Spiegelgasse 1
7 min
500 m
Universität
Basel
Polizeimuseum
Basel-Stadt
Confiserie Bachmann
Turkish Restaurant
Restaurant Pinar
Indian Restaurant
Indian Tandoori Palace
Coffee Store
Nespresso Boutique
Basel Schifflände
Wildt'sches Haus
am Petersplatz
academia Basel
Petersgraben
6 min
400 m
Peterskirchplatz
Peterskirche
Petersplatz
Coop
Toy Store
Franz Carl Weber
Supermarket
Migros
Parkhaus Storchen
Schifflände
Spiga Ristorante
Peterskirchpl.
Jüdisches Museum der
Schweiz / Galerie
Electronics Store
Interdiscount
Martinskirchplatz
Stiftsgasse
Singer (Bar /
Lounge / Club)
GLOBUS Basel
Warenhaus
Weboffice
Marktpl.
Pharmaziemuseum
der Universität Basel
Mövenpick
Brasserie Baselstab
Boutique Tally Weijl
Ängel oder Aff
Marktplatz
Andreasplatz
Restaurant Gifthüttli
Vesalianum
McDonald's
Hoosesaggmuseum
Basel Tattoo Productions
Google

Seam Carving

Applications

- Route planning
- Path planning in games
- robot navigation
- seam carving
- automated planning
- typesetting in TeX
- routing protocols in networks (OSPF, BGP, RIP)
- routing of telecommunication messages
- traffic routing

Source (partially): Network Flows: Theory, Algorithms, and Applications,
R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993

Variants

What are we interested in?

- Single source: from one vertex s to all other vertices
- Single sink: from all vertices to one vertex t
- Source-sink: from vertex s to vertex t
- All pairs: from every vertex to every vertex

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Graph properties

- arbitrary / non-negative / Euclidean weights
- arbitrary / non-negative / no cycles

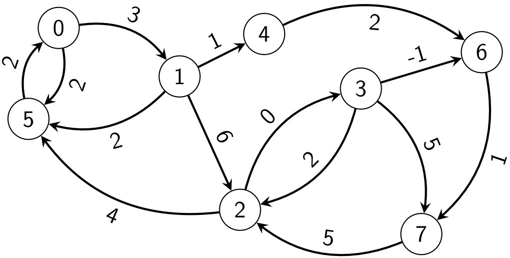
Foundations

Weighted Directed Graphs

Same (high-level) definition of weighted graphs as before, but now we consider directed graphs.

Directed Graph

An (edge)-weighted graph associates every edge e with a weight (or cost) $weight(e) \in \mathbb{R}$.



Reminder: A directed graphs is also called a digraph.

API for Weighted Directed Edge

```
class DirectedEdge:
  # Edge from n1 to n2 with weight w
  def __init__(n1: int, n2: int, w: float) -> None

  # Weight of the edge
  def weight() -> float

  # Initial vertex of the edge
  def from_node() -> int

  # Terminal vertex of the edge
  def to_node() -> int
```

API for Weighted Digraphs

```
class EdgeWeightedDigraph:
  # Graph with no_nodes vertices and no edges
  def __init__(no_nodes: int) -> None

  # Add weighted edge
  def add_edge(e: DirectedEdge) -> None

  # Number of vertices
  def no_nodes() -> int

  # Number of edges
  def no_edges() -> int

  # All outgoing edges of n
  def outgoing_edges(n: int) -> Generator[DirectedEdge]

  # All edges
  def all_edges() -> Generator[DirectedEdge]
```

Shortest Path Problem

Single-source shortest path problem, SSSP

- Given: Graph and start vertex s
- Query for vertex v
  - Is there a path from s to v?
  - If yes, what is the shortest path?

Shortest Path Problem

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- Given: Graph and start vertex s
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  - Is there a path from s to v?
  - If yes, what is the shortest path?
- In weighted graphs:
  Shortest path is the one with lowest weight
  ($=$ minimal sum of edge costs)

API for Shortest-path Implementation

The algorithms for shortest paths should implement the following interface:

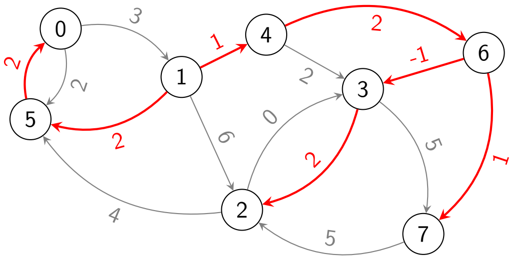
```
class ShortestPaths:
  # Initialization for start vertex s
  def __init__(graph: EdgeWeightedDigraph, s: int) -> None

  # Distance from s to v; infinity, if there is no path
  def dist_to(v: int) -> float

  # Is there a path from s to v?
  def has_path_to(v: int) -> bool

  # Path from s to v; None, if there is none
  def path_to(v: int) -> Generator[DirectedEdge]
```

Shortest-path Tree

Shortest-path Tree

For a weighted digraph G and vertex s, a shortest-path tree is a subgraph that

- forms a directed tree with root s,
- contains all vertices that are reachable from s, and
- for which every path in the tree is a shortest path in G.



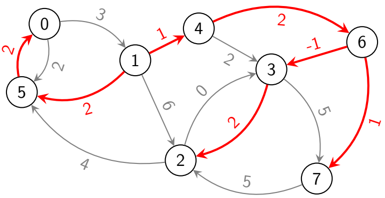
Shortest-path Tree: Representation

Representation: arrays indexed by vertex

- `parent` with reference to parent vertex
  `None` for unreachable vertices and start vertex
- `distance` with distance from the start vertex
  ∞ for unreachable vertices

	0	1	2	3	4	5	6	7
`parent`	5	None	3	6	1	1	4	6

	0	1	2	3	4	5	6	7
`distance`	4	0	4	2	1	2	3	4



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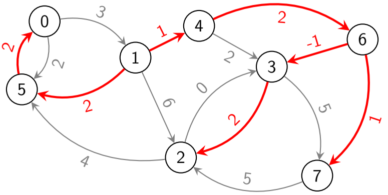

What about parallel edges?

Extracting Shortest Paths

```
    def path_to(self, node):
        if self.distance[node] == float('inf'):
            yield None
        elif self.parent[node] is None:
            yield node
        else:
            # output path from start to parent node
            self.path_to(self.parent[node])
            # finish with node
            yield node
```

This implementation generates a sequence of vertices. What do we have to change to generate a corresponding sequence of edges?

Relaxation

Relaxing edge (u, v)

- `distance[u]`: cost of the shortest known path to u
- `distance[v]`: cost of the shortest known path to v
- `parent[v]`: predecessor of v in the shortest known paths to v
- Does edge (u, v) establish a shorter path to v (through u)?
- If yes, update `distance[v]` and `parent[v]`.

Illustration: Whiteboard

Relaxation

```python
def relax(self, edge):
    u = edge.from_node()
    v = edge.to_node()
    if self.distance[v] > self.distance[u] + edge.weight():
        self.parent[v] = u
        self.distance[v] = self.distance[u] + edge.weight()
```

Optimality Criterion and Generic Algorithm

Optimality Criterion

Theorem

Let G be a weighted digraph without negative cycles.
Array `distance[]` contains the cost of the shortest paths from s if and only if

1. `distance[s]` $= 0$
2. $\texttt{distance}[w] \leq \texttt{distance}[v] + weight(e)$
   for all edges $e = (v, w)$, and
3. *for all vertices v, $\texttt{distance}[v]$ is the cost of some path from s to v, or ∞ if there is no such path.*

Optimality Criterion (Continued)

Proof

"$\Rightarrow$"

1. Since the graph has no cycles of negative cost, no path from s to s can have negative cost. Thus, the empty path is optimal and `distance[s]` is 0.

Optimality Criterion (Continued)

Proof

"$\Rightarrow$"

1. Since the graph has no cycles of negative cost, no path from s to s can have negative cost. Thus, the empty path is optimal and `distance[s]` is 0.
2. Consider an arbitrary edge e from u to v.

   The shortest path from s to u has cost `distance[u]`. If we extend this path by edge e, we have a path from s to v of cost `distance[u]` $+$ *weight*(e). Thus, the cost of a shortest path from s to v cannot be larger and it holds that `distance[v]` $\leq$ `distance[u]` $+$ *weight*(e).

Optimality Criterion (Continued)

Proof

"$\Rightarrow$"

1. Since the graph has no cycles of negative cost, no path from s to s can have negative cost. Thus, the empty path is optimal and `distance[s]` is 0.
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   The shortest path from s to u has cost `distance[u]`. If we extend this path by edge e, we have a path from s to v of cost `distance[u]` + $weight(e)$. Thus, the cost of a shortest path from s to v cannot be larger and it holds that `distance[v]` $\leq$ `distance[u]` + $weight(e)$.
3. Trivially true.

...

Optimality Criterion (Continued)

Proof (continued).

"$\Leftarrow$"

For unreachable vertices, the value is infinity by definition.

Consider an arbitrary vertex v and a shortest path $p = (v_0, \ldots, v_n)$ from s to v, i.e. $v_0 = s$, $v_n = v$.
For $i \in \{1, \ldots, n\}$, let e_i be a cheapest edge from v_{i-1} to v_i.
Since all inequalities are satisfied, we have

$$\begin{aligned}
\texttt{distance}[v_n] &\leq \texttt{distance}[v_{n-1}] + \mathit{weight}(e_n) \\
&\leq \texttt{distance}[v_{n-2}] + \mathit{weight}(e_{n-1}) + \mathit{weight}(e_n) \\
\leq \ldots &\leq \mathit{weight}(e_1) + \cdots + \mathit{weight}(e_n) \\
&= \text{ cost of an optimal path.}
\end{aligned}$$

Due to 3, $\texttt{distance}[v_n]$ cannot be lower than the optimal cost. $\square$

Generic Algorithm

Generic Algorithm for Start Vertex s

- Initialize $\texttt{distance}[s] = 0$ and $\texttt{distance}[v] = \infty$ for all other vertices
- As long as the optimality criterion is not satisfied: Relax an arbitrary edge

Correct:

- Finite `distance[v]` always corresponds to the cost of a path from s to v.
- Every successful relaxation reduces `distance[v]` for some v.
- For every vertex, the distance can only be reduced finitely often.

Summary

Summary

- Single-source shortest paths: Compute in a weighted digraph the shortest paths from a given vertex to all reachable vertices.
- Relaxation: If for edge (u,v) the best known distance to v is larger than the one to u plus the edge cost, then update the distance to v (with predecessor u).
- Generic algorithm
  - Based on relaxation and optimality criterion.
  - Every instantiation is correct for all weighted digraphs without negative-cost cycles.
  - Specific instantiations: next chapter.