Algorithms and Data Structures
C5. Shortest Paths: Foundations

Gabriele Röger

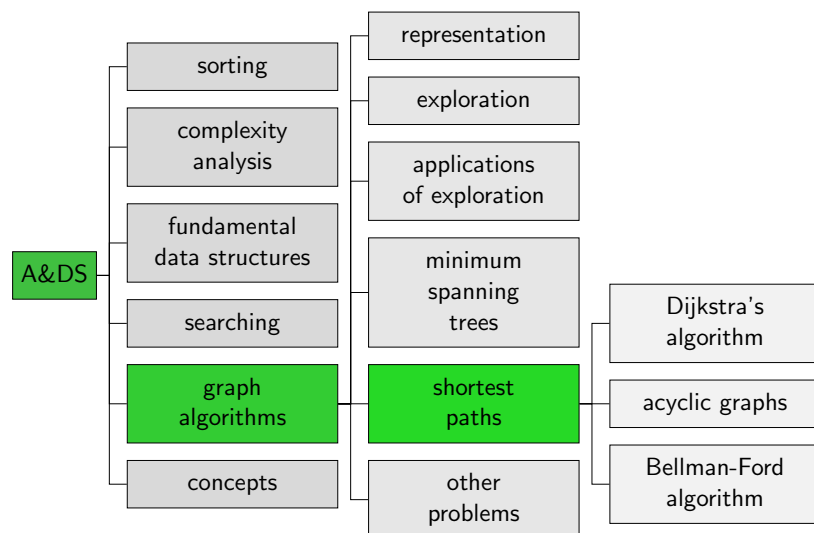University of Basel

May 22, 2024

C5.1 Introduction

C5.2 Foundations

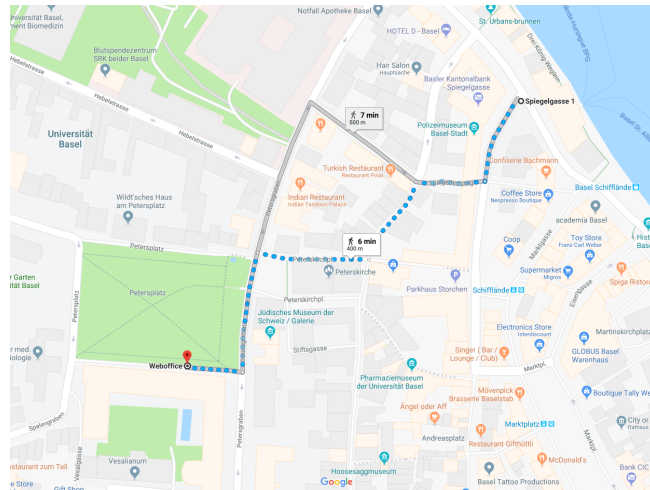C5.3 Optimality Criterion and Generic Algorithm

C5.4 Summary

Content of the Course

C5.1 Introduction

Google Maps

Seam Carving

Applications

- ▶ Route planning
- ▶ Path planning in games
- ▶ robot navigation
- ▶ seam carving
- ▶ automated planning
- ▶ typesetting in TeX
- ▶ routing protocols in networks (OSPF, BGP, RIP)
- ▶ routing of telecommunication messages
- ▶ traffic routing

Source (partially): Network Flows: Theory, Algorithms, and Applications,
         R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993

Variants

What are we interested in?
- ▶ Single source: from one vertex s to all other vertices
- ▶ Single sink: from all vertices to one vertex t
- ▶ Source-sink: from vertex s to vertex t
- ▶ All pairs: from every vertex to every vertex

Graph properties
- ▶ arbitrary / non-negative / Euclidean weights
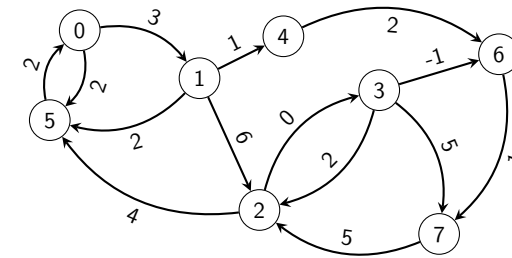- ▶ arbitrary / non-negative / no cycles

C5.2 Foundations

Weighted Directed Graphs

Same (high-level) definition of weighted graphs as before, but now we consider directed graphs.

> **Directed Graph**
> An (edge)-weighted graph associates every edge e with a weight (or cost) $weight(e) \in \mathbb{R}$.

Reminder: A directed graphs is also called a digraph.

API for Weighted Directed Edge

```python
class DirectedEdge:
    # Edge from n1 to n2 with weight w
    def __init__(n1: int, n2: int, w: float) -> None

    # Weight of the edge
    def weight() -> float

    # Initial vertex of the edge
    def from_node() -> int

    # Terminal vertex of the edge
    def to_node() -> int
```

API for Weighted Digraphs

```python
class EdgeWeightedDigraph:
    # Graph with no_nodes vertices and no edges
    def __init__(no_nodes: int) -> None

    # Add weighted edge
    def add_edge(e: DirectedEdge) -> None

    # Number of vertices
    def no_nodes() -> int

    # Number of edges
    def no_edges() -> int

    # All outgoing edges of n
    def outgoing_edges(n: int) -> Generator[DirectedEdge]

    # All edges
    def all_edges() -> Generator[DirectedEdge]
```

Shortest Path Problem

Single-source shortest path problem, SSSP
- Given: Graph and start vertex s
- Query for vertex v
  - Is there a path from s to v?
  - If yes, what is the shortest path?
- In weighted graphs:
  Shortest path is the one with lowest weight
  ($=$ minimal sum of edge costs)

API for Shortest-path Implementation

The algorithms for shortest paths should implement the following interface:

```
1 class ShortestPaths:
2    # Initialization for start vertex s
3    def __init__(graph: EdgeWeightedDigraph, s: int) -> None
4
5    # Distance from s to v; infinity, if there is no path
6    def dist_to(v: int) -> float
7
8    # Is there a path from s to v?
9    def has_path_to(v: int) -> bool
10
11   # Path from s to v; None, if there is none
12   def path_to(v: int) -> Generator[DirectedEdge]
```

Shortest-path Tree

Shortest-path Tree
For a weighted digraph G and vertex s, a shortest-path tree is a subgraph that
- forms a directed tree with root s,
- contains all vertices that are reachable from s, and
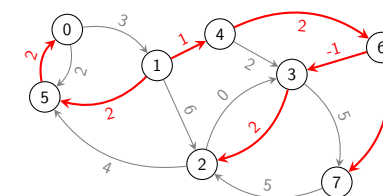- for which every path in the tree is a shortest path in G.
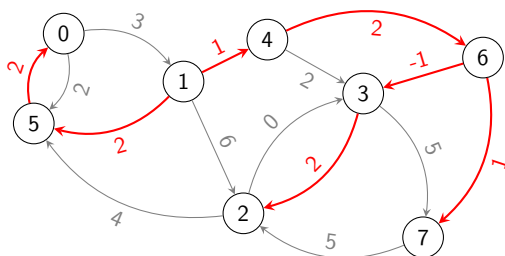
Shortest-path Tree: Representation

Representation: arrays indexed by vertex
- `parent` with reference to parent vertex
  `None` for unreachable vertices and start vertex
- `distance` with distance from the start vertex
  ∞ for unreachable vertices

parent	0	1	2	3	4	5	6	7
	5	None	3	6	1	1	4	6

distance	0	1	2	3	4	5	6	7
	4	0	4	2	1	2	3	4

What about parallel edges?

Extracting Shortest Paths

```python
1    def path_to(self, node):
2        if self.distance[node] == float('inf'):
3            yield None
4        elif self.parent[node] is None:
5            yield node
6        else:
7            # output path from start to parent node
8            self.path_to(self.parent[node])
9            # finish with node
10           yield node
```

This implementation generates a sequence of vertices. What do we have to change to generate a corresponding sequence of edges?

Relaxation

Relaxing edge (u, v)
- ▶ distance[u]: cost of the shortest known path to u
- ▶ distance[v]: cost of the shortest known path to v
- ▶ parent[v]: predecessor of v in the shortest known paths to v
- ▶ Does edge (u, v) establish a shorter path to v (through u)?
- ▶ If yes, update distance[v] and parent[v].

Illustration: Whiteboard

Relaxation

```python
1    def relax(self, edge):
2        u = edge.from_node()
3        v = edge.to_node()
4        if self.distance[v] > self.distance[u] + edge.weight():
5            self.parent[v] = u
6            self.distance[v] = self.distance[u] + edge.weight()
```

C5.3 Optimality Criterion and Generic Algorithm

Optimality Criterion

> **Theorem**
> *Let G be a weighted digraph without negative cycles.*
> *Array `distance[]` contains the cost of the shortest paths from s*
> *if and only if*
>
> ❶ `distance[s] = 0`
> ❷ `distance[w] ≤ distance[v] + weight(e)`
>   *for all edges e = (v, w), and*
> ❸ *for all vertices v, `distance[v]` is the cost of some path*
>   *from s to v, or ∞ if there is no such path.*

Optimality Criterion (Continued)

> **Proof**
> "⇒"
>
> ❶ Since the graph has no cycles of negative cost, no path from s
>   to s can have negative cost. Thus, the empty path is optimal
>   and `distance[s]` is 0.
> ❷ Consider an arbitrary edge e from u to v.
>
>   The shortest path from s to u has cost `distance[u]`. If we
>   extend this path by edge e, we have a path from s to v of
>   cost `distance[u]` $+ weight(e)$. Thus, the cost of a shortest
>   path from s to v cannot be larger and it holds that
>   `distance[v]` $\leq$ `distance[u]` $+ weight(e)$.
> ❸ Trivially true.
>
>   . . .

Optimality Criterion (Continued)

> **Proof (continued).**
> "⇐"
> For unreachable vertices, the value is infinity by definition.
>
> Consider an arbitrary vertex v and a shortest path $p = (v_0, \ldots, v_n)$
> from s to v, i.e. $v_0 = s$, $v_n = v$.
> For $i \in \{1, \ldots, n\}$, let e_i be a cheapest edge from v_{i-1} to v_i.
> Since all inequalities are satisfied, we have
>
> $$\begin{aligned} \texttt{distance}[v_n] &\leq \texttt{distance}[v_{n-1}] + weight(e_n) \\ &\leq \texttt{distance}[v_{n-2}] + weight(e_{n-1}) + weight(e_n) \\ &\leq \ldots \leq weight(e_1) + \cdots + weight(e_n) \\ &= \text{ cost of an optimal path.} \end{aligned}$$
>
> Due to 3, `distance`$[v_n]$ cannot be lower than the optimal cost. □

Generic Algorithm

> **Generic Algorithm for Start Vertex s**
> ▶ Initialize `distance`$[s] = 0$ and
>   `distance`$[v] = \infty$ for all other vertices
> ▶ As long as the optimality criterion is not satisfied:
>   Relax an arbitrary edge

Correct:

▶ Finite `distance[v]` always corresponds to the cost of a path
  from s to v.
▶ Every successful relaxation reduces `distance[v]` for some v.
▶ For every vertex, the distance can only be reduced finitely
  often.

C5.4 Summary

Summary

▶ **Single-source shortest paths**: Compute in a weighted digraph the shortest paths from a given vertex to all reachable vertices.

▶ **Relaxation**: If for edge (u,v) the best known distance to v is larger than the one to u plus the edge cost, then update the distance to v (with predecessor u).

▶ **Generic algorithm**
  ▶ Based on relaxation and optimality criterion.
  ▶ Every instantiation is correct for all weighted digraphs without negative-cost cycles.
  ▶ Specific instantiations: next chapter.