Algorithms and Data Structures

C2. Graph Exploration: Applications

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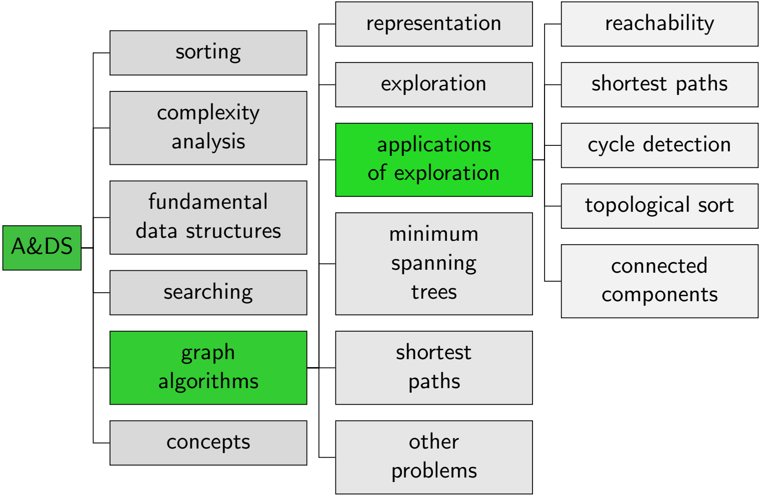
Reminder: Graph Exploration

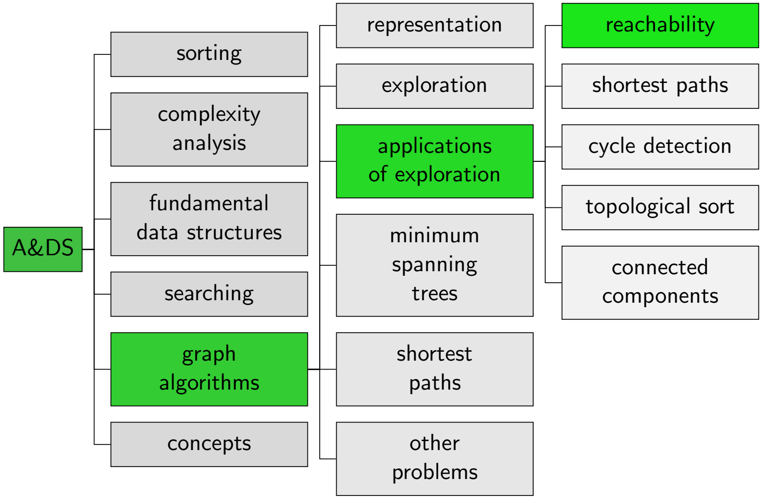
- Given a vertex v, visit all vertices that are reachable from v.
- Often used as part of other graph algorithms.
- Depth-first search: go "deep" into the graph (away from v)
- Breadth-first search: first all neighbours, then neighbours of neighbours, ...

Content of the Course



Reachability

Content of the Course



Mark-and-Sweep Garbage Collection

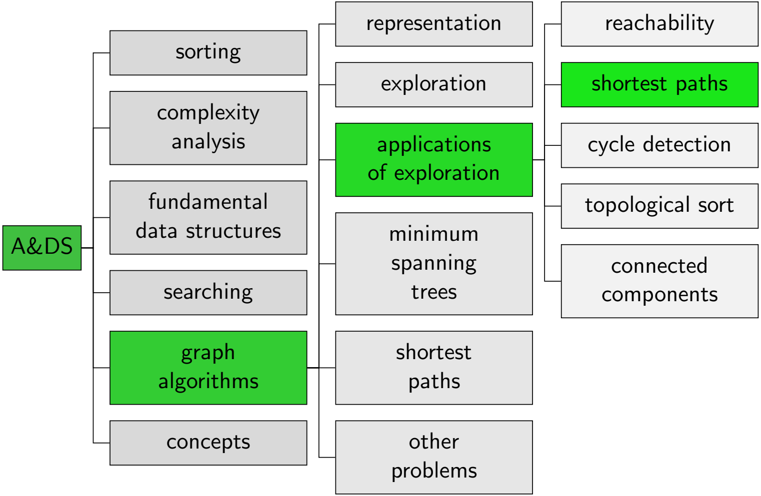
Aim: Release memory occupied by no longer accessible objects.

- Directed graph: Objects as vertices, references to objects as edges.
- One bit per object for marker during garbage collection.
- Mark: Mark all reachable objects (set bit to 1).
- Sweep: Clear unmarked objects from memory.
  Afterwards set bit for all reachable objects back to 0.

Magic Wand in Image Editing

Shortest Paths

Content of the Course



Shortest Paths: Idea

- Breadth-first search visits the vertices with increasing (minimal) distance from the start vertex.
- First visit of a vertex happens on shortest path.
- Idea: Use path from induced search tree.

Jupyter Notebook

Jupyter notebook: `graph_exploration_applications.ipynb`

Shortest-path Problem

Single-source Shortest-paths Problem

- Given: Graph and start vertex s
- Query for vertex v
  - Is there a path from s to v?
  - If yes, what is the shortest path?
- Abbreviation SSSP

Shortest Paths: Algorithm

```
class SingleSourceShortestPaths:
    def __init__(self, graph, start_node):
        self.predecessor = [None] * graph.no_nodes()
        self.predecessor[start_node] = start_node

        # precompute predecessors with breadth-first search with
        # self.predecessors used for detecting visited nodes
        queue = deque()
        queue.append(start_node)
        while queue:
            v = queue.popleft()
            for s in graph.successors(v):
                if self.predecessor[s] is None:
                    self.predecessor[s] = v
                    queue.append(s)
    ...
```

In principle as before
(just as a class)

Shortest Paths: Algorithm (Continued)

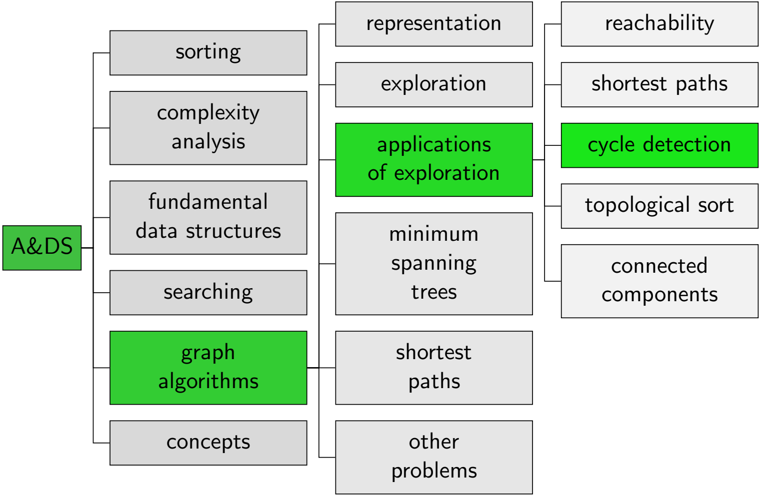
```
    def has_path_to(self, node):
        return self.predecessor[node] is not None

    def get_path_to(self, node):
        if not self.has_path_to(node):
            return None
        if self.predecessor[node] == node:  # start node
            return [node]
        pre = self.predecessor[node]
        path = self.get_path_to(pre)
        path.append(node)
        return path
```

Running time?

Later: Shortest paths with edge weights

Acyclic Graphs

Content of the Course



Detection of Acyclic Graphs

Definition (Directed Acyclic Graph)

A directed acyclic graph (DAG) is a directed graph that contains no directed cycles.

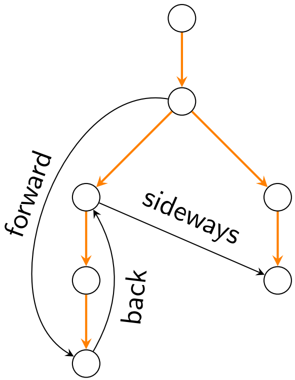
Detection of Acyclic Graphs

Definition (Directed Acyclic Graph)

A directed acyclic graph (DAG) is a directed graph that contains no directed cycles.

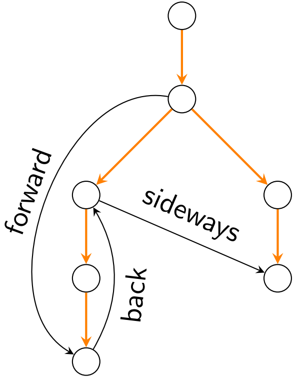
Task: Decide whether a directed graph contains a cycle. If yes, return a cycle.

Criterion for Acyclicity



Induced search tree of a depth-first search (orange) and possible other edges

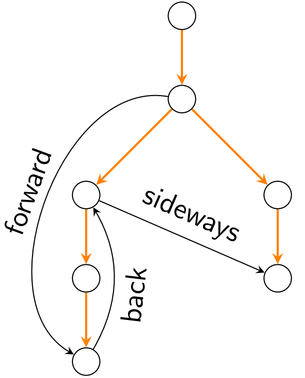
Criterion for Acyclicity



Induced search tree of a depth-first search (orange) and possible other edges

The (reachable part of the) graph is acyclic if and only if there are no back edges.

Criterion for Acyclicity



Induced search tree of a depth-first search (orange) and possible other edges

The (reachable part of the) graph is acyclic if and only if there are no back edges.

Idea: Remember the vertices on the current path in a DFS.

Cycle Detection: Algorithm

```
class DirectedCycle:
    def __init__(self, graph):
        self.predecessor = [None] * graph.no_nodes()
        self.on_current_path = [False] * graph.no_nodes()
        self.cycle = None
        for node in range(graph.no_nodes()):
            if self.has_cycle():
                break
            if self.predecessor[node] is None:
                self.predecessor[node] = node
                self.dfs(graph, node)

    def has_cycle(self):
        return self.cycle is not None
```

Repeated depth-first searches such that at the end all vertices have been visited.

Cycle Detection: Algorithm (Continued)

```
    def dfs(self, graph, node):
        self.on_current_path[node] = True
        for s in graph.successors(node):
            if self.has_cycle():
                return
            if self.on_current_path[s]:
                self.predecessor[s] = node
                self.extract_cycle(s)
            if self.predecessor[s] is None:
                self.predecessor[s] = node
                self.dfs(graph, s)
        self.on_current_path[node] = False
```

Cycle Detection: Algorithm (Continued)

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    def dfs(self, graph, node):
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            if self.predecessor[s] is None:
                self.predecessor[s] = node
                self.dfs(graph, s)
        self.on_current_path[node] = False
```

Update whether vertex is on the current path.

Cycle Detection: Algorithm (Continued)

```
    def dfs(self, graph, node):
        self.on_current_path[node] = True
        for s in graph.successors(node):
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            if self.predecessor[s] is None:
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                self.dfs(graph, s)
        self.on_current_path[node] = False
```

Found a cycle

Update whether vertex is on the current path.

Cycle Detection: Algorithm (Continued)

```
    def dfs(self, graph, node):
        self.on_current_path[node] = True
        for s in graph.successors(node):
            if self.has_cycle():
                return
            if self.on_current_path[s]:
                self.predecessor[s] = node
                self.extract_cycle(s)
            if self.predecessor[s] is None:
                self.predecessor[s] = node
                self.dfs(graph, s)
        self.on_current_path[node] = False
```

Skip if a cycle has been detected somewhere.

Found a cycle

Update whether vertex is on the current path.

Cycle Detection: Algorithm (Continued)

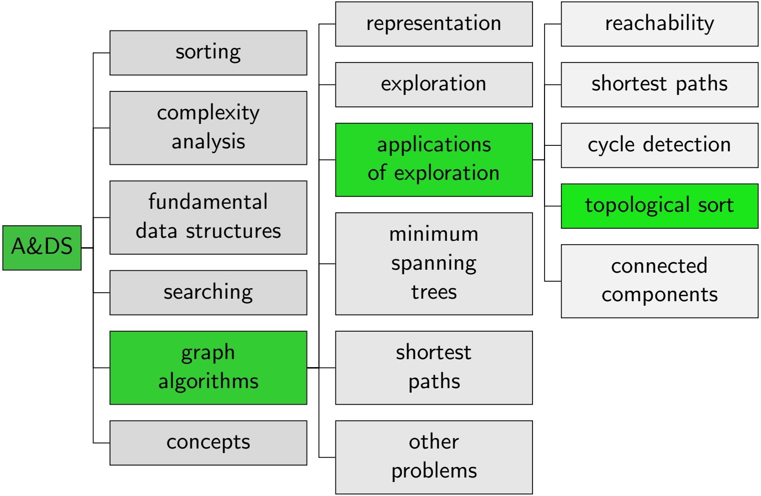
When calling `extract_cycle`, `node` is on a cycle in `self.predecessor`.

```
    def extract_cycle(self, node):
        self.cycle = deque()
        current = node
        self.cycle.appendleft(current)
        while True:
            current = self.predecessor[current]
            self.cycle.appendleft(current)
            if current == node:
                return
```

Jupyter Notebook

Jupyter notebook: `graph_exploration_applications.ipynb`

Content of the Course



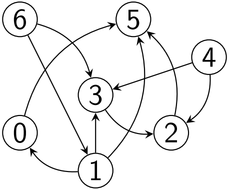
Topological Sort

Definition

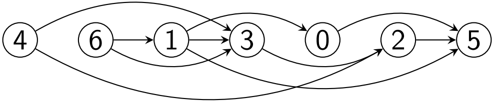
A topological sort of a directed acyclic graph $G = (V, E)$ is a linear ordering of all its vertices such that if G contains an edge (u, v), then u appears before v in the ordering.

For example relevant for scheduling:
edge (u, v) expresses that job u must be completed before job v can be started.

Topological Sort: Illustration



Topological sort: 4, 6, 1, 3, 0, 2, 5



Topological Sort: Algorithm

Theorem

For the reachable part of a acyclic graph, the reverse DFS postorder is a topological sort.

Algorithm:

- Sequence of depth-first searches (for still unvisited vertices) until all vertices visited.
- Store for each DFS the reverse postorder: P_i for i-th search
- Let k be the number of searches. Then the concatenation $P_k, \ldots, P_1$ is a topological sort.

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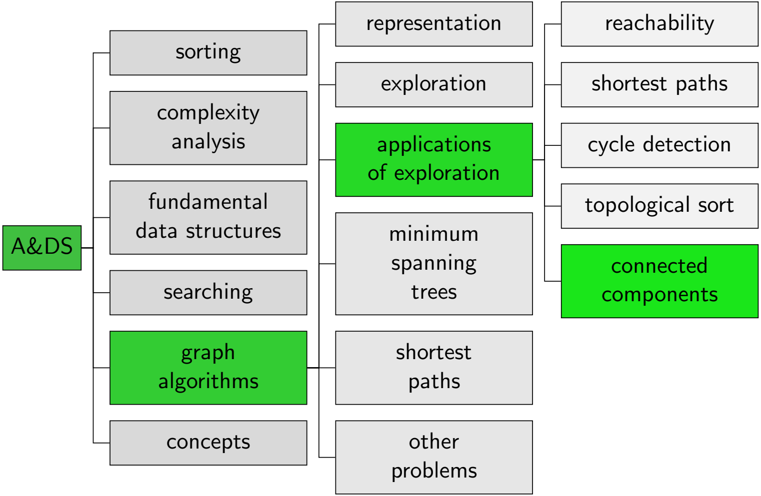
Connected Components

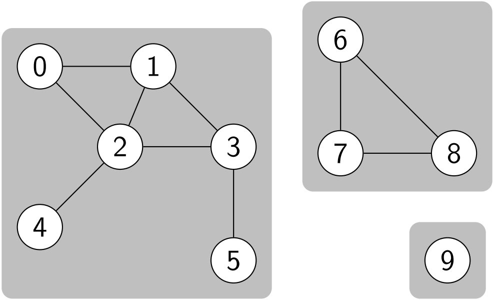
Content of the Course



Connected Components of Undirected Graphs

Undirected graph

- Two vertices u and v are in the same connected component if there is a path between u and v.



Connected Components: Interface

We want to implement the following interface:

```
class ConnectedComponents:
  # Initialization with precomputation
  def __init__(graph: UndirectedGraph) -> None

  # Are vertices node1 and node2 connected?
  def connected(node1: int, node2: int) -> bool

  # Number of connected components
  def count() -> int

  # Component number for node
  # (between 0 and count()-1)
  def id(node: int) -> int
```

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  # Component number for node
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  def id(node: int) -> int
```

Idea: Sequence of graph explorations until all vertices visited.
ID of vertex corresponds to iteration in which it was visited.

Connected Components: Algorithm

```python
class ConnectedComponents:
    def __init__(self, graph):
        self.id = [None] * graph.no_nodes()
        self.curr_id = 0
        visited = [False] * graph.no_nodes()
        for node in range(graph.no_nodes()):
            if not visited[node]:
                self.dfs(graph, node, visited)
                self.curr_id += 1

    def dfs(self, graph, node, visited):
        if visited[node]:
            return
        visited[node] = True
        self.id[node] = self.curr_id
        for n in graph.neighbours(node):
            self.dfs(graph, n, visited)
```

How are connected, count and id implemented?

Jupyter Notebook

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Connected Components of Directed Graphs

Directed graph G

- If one ignores the arc directions, then every connected component of the resulting undirected graph is a weakly connected component of G.

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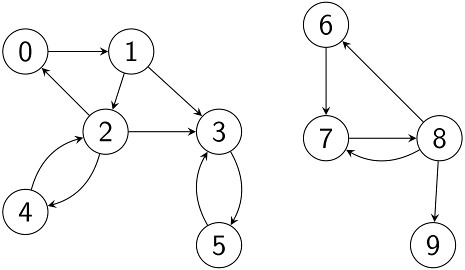
Connected Components of Directed Graphs

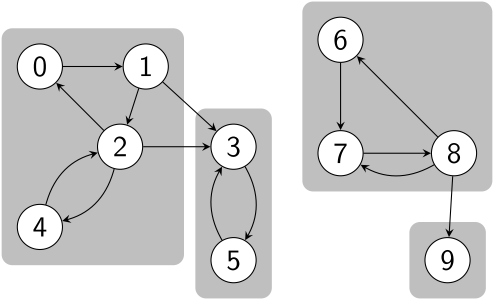
Directed graph G

- If one ignores the arc directions, then every connected component of the resulting undirected graph is a weakly connected component of G.
- G is strongly connected, if there is a directed path from each vertex to each other vertex.
- A strongly connected component of G is a maximal strongly connected subgraph.

Strongly Connected Components



Strongly Connected Components



Strongly Connected Components

Kosaraju' algorithm

- Given directed graph $G = (V, E)$, compute a reverse postorder P (for all vertices) of the graph $G^R = (V, \{(v, u) \mid (u, v) \in E\})$ (all edges reversed).
- Conduct a sequence of explorations in G, always selecting the first still unvisited vertex in P as the next start vertex.
- All vertices that are reached by the same exploration, are in the same strongly connected component.

Jupyter Notebook

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Summary

Summary

We have seen a number of applications of graph exploration:

- Reachability
- Shortest paths
- Cycle detection
- Topological sort
- Connected components

Some applications require a specific exploration, for other applications we can use both, BFS and DFS.