Algorithms and Data Structures

C2. Graph Exploration: Applications

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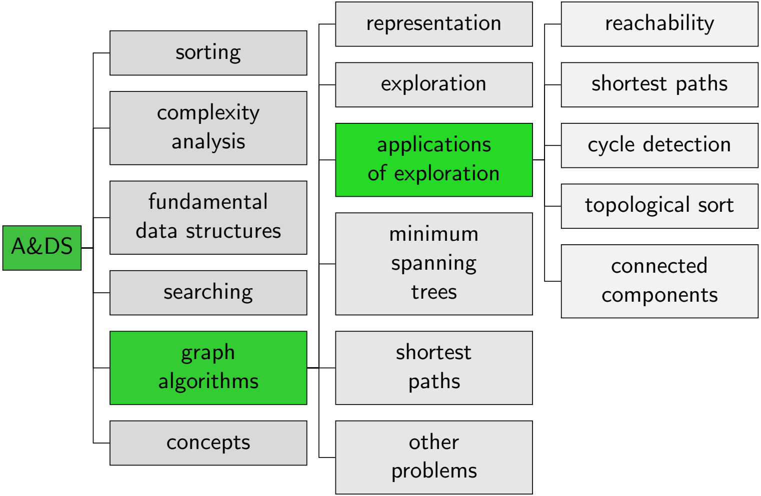
Reminder: Graph Exploration

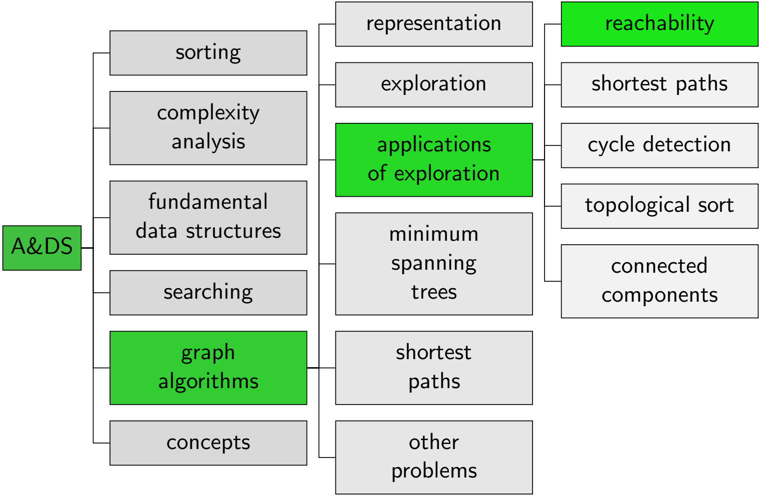
- Given a vertex v, visit all vertices that are reachable from v.
- Often used as part of other graph algorithms.
- Depth-first search: go “deep” into the graph (away from v)
- Breadth-first search: first all neighbours, then neighbours of neighbours, ...

Content of the Course



C2.1 Reachability

Content of the Course



Mark-and-Sweep Garbage Collection

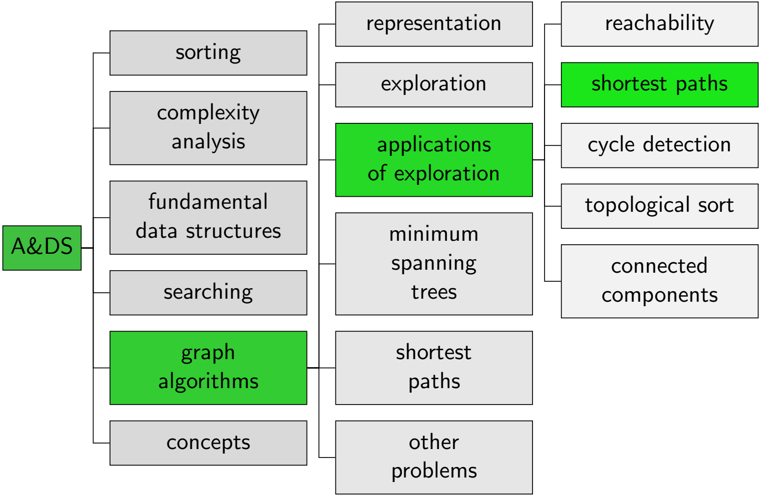
Aim: Release memory occupied by no longer accessible objects.

- Directed graph: Objects as vertices, references to objects as edges.
- One bit per object for marker during garbage collection.
- Mark: Mark all reachable objects (set bit to 1).
- Sweep: Clear unmarked objects from memory.
  Afterwards set bit for all reachable objects back to 0.

Magic Wand in Image Editing

C2.2 Shortest Paths

Content of the Course



Shortest Paths: Idea

- Breadth-first search visits the vertices with increasing (minimal) distance from the start vertex.
- First visit of a vertex happens on shortest path.
- Idea: Use path from induced search tree.

Jupyter Notebook

Jupyter notebook: `graph_exploration_applications.ipynb`

Shortest-path Problem

Single-source Shortest-paths Problem

- Given: Graph and start vertex s
- Query for vertex v
  - Is there a path from s to v?
  - If yes, what is the shortest path?
- Abbreviation SSSP

Shortest Paths: Algorithm

```
class SingleSourceShortestPaths:
    def __init__(self, graph, start_node):
        self.predecessor = [None] * graph.no_nodes()
        self.predecessor[start_node] = start_node

        # precompute predecessors with breadth-first search with
        # self.predecessors used for detecting visited nodes
        queue = deque()
        queue.append(start_node)
        while queue:
            v = queue.popleft()
            for s in graph.successors(v):
                if self.predecessor[s] is None:
                    self.predecessor[s] = v
                    queue.append(s)
    ...
```

In principle as before
(just as a class)

Shortest Paths: Algorithm (Continued)

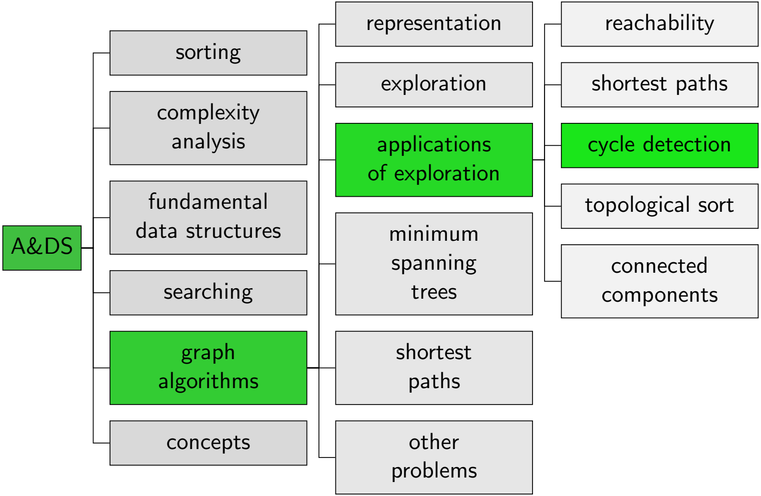
```python
    def has_path_to(self, node):
        return self.predecessor[node] is not None

    def get_path_to(self, node):
        if not self.has_path_to(node):
            return None
        if self.predecessor[node] == node:  # start node
            return [node]
        pre = self.predecessor[node]
        path = self.get_path_to(pre)
        path.append(node)
        return path
```

Running time?

Later: Shortest paths with edge weights

C2.3 Acyclic Graphs

Content of the Course



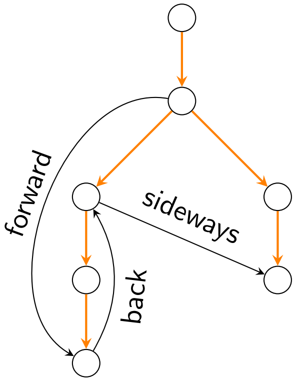
Detection of Acyclic Graphs

Definition (Directed Acyclic Graph)

A directed acyclic graph (DAG) is a directed graph that contains no directed cycles.

Task: Decide whether a directed graph contains a cycle. If yes, return a cycle.

Criterion for Acyclicity



Induced search tree of a depth-first search (orange) and possible other edges

The (reachable part of the) graph is acyclic if and only if there are no back edges.

Idea: Remember the vertices on the current path in a DFS.

Cycle Detection: Algorithm

```
class DirectedCycle:
    def __init__(self, graph):
        self.predecessor = [None] * graph.no_nodes()
        self.on_current_path = [False] * graph.no_nodes()
        self.cycle = None
        for node in range(graph.no_nodes()):
            if self.has_cycle():
                break
            if self.predecessor[node] is None:
                self.predecessor[node] = node
                self.dfs(graph, node)

    def has_cycle(self):
        return self.cycle is not None
```

Repeated depth-first searches such that at the end all vertices have been visited.

Cycle Detection: Algorithm (Continued)

```
    def dfs(self, graph, node):
        self.on_current_path[node] = True
        for s in graph.successors(node):
            if self.has_cycle():
                return
            if self.on_current_path[s]:
                self.predecessor[s] = node
                self.extract_cycle(s)
            if self.predecessor[s] is None:
                self.predecessor[s] = node
                self.dfs(graph, s)
        self.on_current_path[node] = False
```

Skip if a cycle has been detected somewhere.

Found a cycle

Update whether vertex is on the current path.

Cycle Detection: Algorithm (Continued)

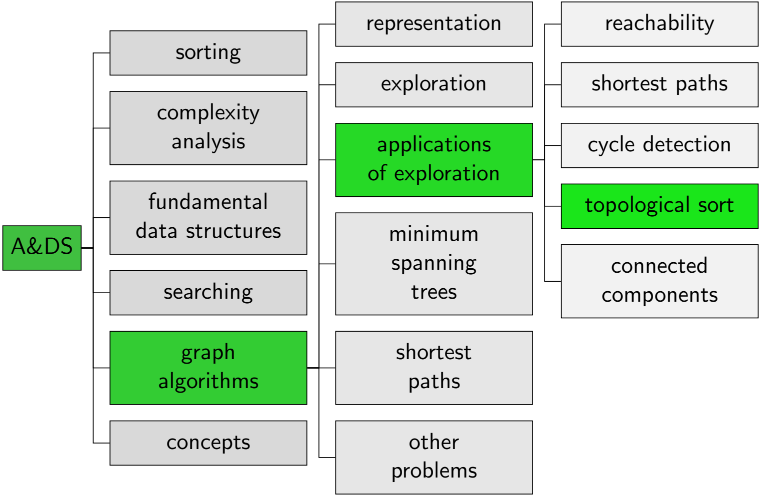
When calling `extract_cycle`, `node` is on a cycle in `self.predecessor`.

```
    def extract_cycle(self, node):
        self.cycle = deque()
        current = node
        self.cycle.appendleft(current)
        while True:
            current = self.predecessor[current]
            self.cycle.appendleft(current)
            if current == node:
                return
```

Jupyter Notebook

Jupyter notebook: `graph_exploration_applications.ipynb`

Content of the Course



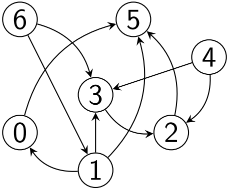
Topological Sort

Definition

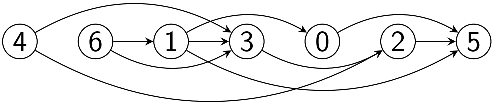
A topological sort of a directed acyclic graph $G = (V, E)$ is a linear ordering of all its vertices such that if G contains an edge (u, v), then u appears before v in the ordering.

For example relevant for scheduling:
edge (u, v) expresses that job u must be completed before job v can be started.

Topological Sort: Illustration



Topological sort: 4, 6, 1, 3, 0, 2, 5



Topological Sort: Algorithm

Theorem

For the reachable part of a acyclic graph, the reverse DFS postorder is a topological sort.

Algorithm:

- Sequence of depth-first searches (for still unvisited vertices) until all vertices visited.
- Store for each DFS the reverse postorder:
  P_i for i-th search
- Let k be the number of searches. Then the concatenation $P_k, \ldots, P_1$ is a topological sort.

Jupyter Notebook

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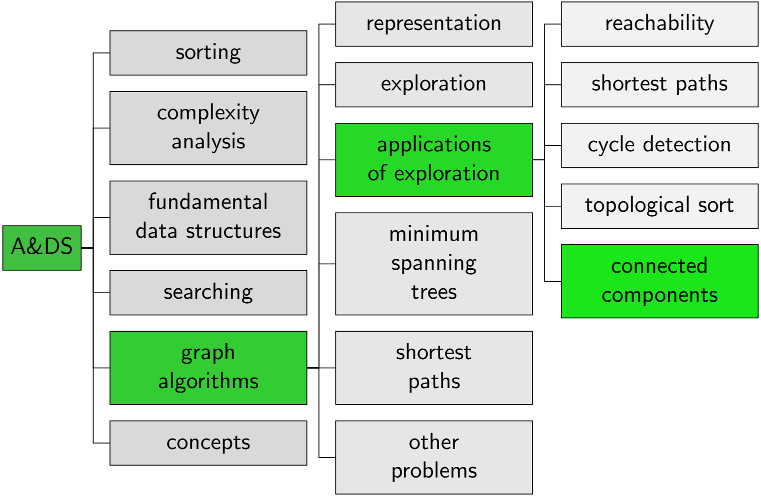
C2.4 Connected Components

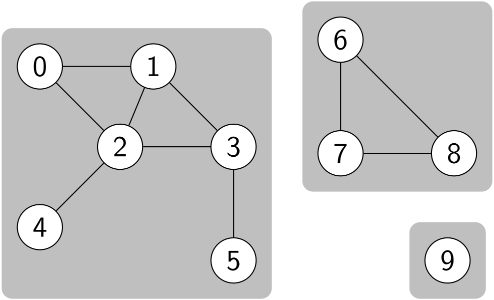
Content of the Course



Connected Components of Undirected Graphs

Undirected graph

- Two vertices u and v are in the same connected component if there is a path between u and v.



Connected Components: Interface

We want to implement the following interface:

```
class ConnectedComponents:
    # Initialization with precomputation
    def __init__(graph: UndirectedGraph) -> None

    # Are vertices node1 and node2 connected?
    def connected(node1: int, node2: int) -> bool

    # Number of connected components
    def count() -> int

    # Component number for node
    # (between 0 and count()-1)
    def id(node: int) -> int
```

Idea: Sequence of graph explorations until all vertices visited.
ID of vertex corresponds to iteration in which it was visited.

Connected Components: Algorithm

```python
class ConnectedComponents:
    def __init__(self, graph):
        self.id = [None] * graph.no_nodes()
        self.curr_id = 0
        visited = [False] * graph.no_nodes()
        for node in range(graph.no_nodes()):
            if not visited[node]:
                self.dfs(graph, node, visited)
                self.curr_id += 1

    def dfs(self, graph, node, visited):
        if visited[node]:
            return
        visited[node] = True
        self.id[node] = self.curr_id
        for n in graph.neighbours(node):
            self.dfs(graph, n, visited)
```

How are connected, count and id implemented?

Jupyter Notebook

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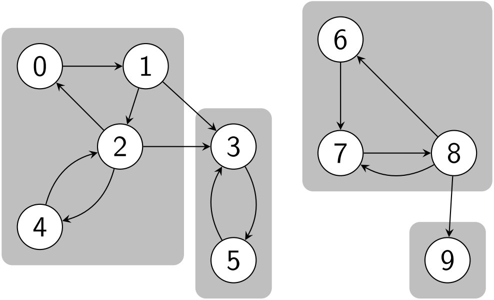
Connected Components of Directed Graphs

Directed graph G

- If one ignores the arc directions, then every connected component of the resulting undirected graph is a weakly connected component of G.
- G is strongly connected, if there is a directed path from each vertex to each other vertex.
- A strongly connected component of G is a maximal strongly connected subgraph.

Strongly Connected Components



Strongly Connected Components

Kosaraju' algorithm

- Given directed graph $G = (V, E)$, compute a reverse postorder P (for all vertices) of the graph $G^R = (V, \{(v, u) \mid (u, v) \in E\})$ (all edges reversed).
- Conduct a sequence of explorations in G, always selecting the first still unvisited vertex in P as the next start vertex.
- All vertices that are reached by the same exploration, are in the same strongly connected component.

Jupyter Notebook

Jupyter notebook: `graph_exploration_applications.ipynb`

C2.5 Summary

Summary

We have seen a number of applications of graph exploration:

- Reachability
- Shortest paths
- Cycle detection
- Topological sort
- Connected components

Some applications require a specific exploration, for other applications we can use both, BFS and DFS.