Algorithms and Data Structures

B6. Red-Black Trees

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Red-Black Trees

Motivation

- Binary search trees can support many relevant operations in linear time in the height of the tree.
- But: Binary search trees can degenerate into chains, in which case the operations take linear time in the number n of elements (no better than with a linked list).
- Idea: Search-trees schemes that are in some form "balanced" and can guarantee running time $O(\log_2 n)$ in the worst case.
  - AVL trees: for every node, the height of the left and right subtree differs by at most 1.
  - B-trees: permit several keys and subtrees per node (e.g. special case: 2-3 tree).
  - Red-Black trees: use node colors to maintain an approximate balancing.
  - ...

Red-Black Trees: Representation

- Use one extra bit per node, storing its color, which can be either red or black.
- Each node now contains attributes color, key, left, right and parent.

`None` Leaf Nodes

- Left, right and parent are `None` if there is no corresponding node.

None Leaf Nodes

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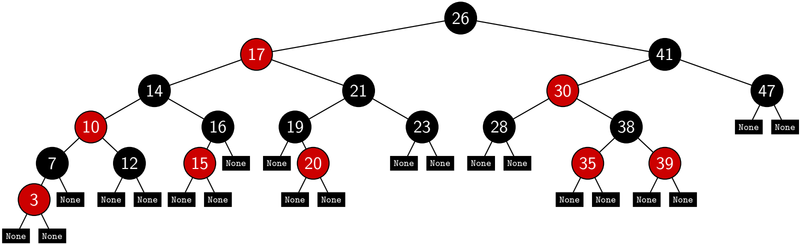
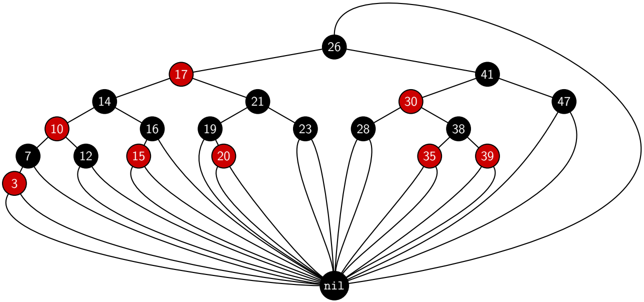
- Because it is conceptionally and implementation-wise easier, we will represent them as actual node objects.
- These are then the leaves of the trees and the nodes holding the entries are inner nodes.



None Leaf Nodes: Sentinel

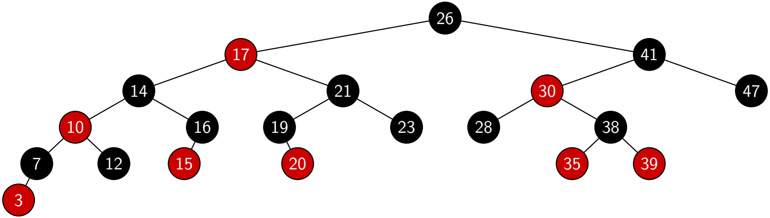
Instead of many leaf nodes, we use a single sentinel node `nil`.

- Implemented like a normal (black) node
  but used as child of many nodes.
- The sentinel also serves as parent of the root.
- Attributes for parent and children can take on arbitrary values.



Graphical Representation

On the slides, we omit the `None` leaf nodes/sentinel:



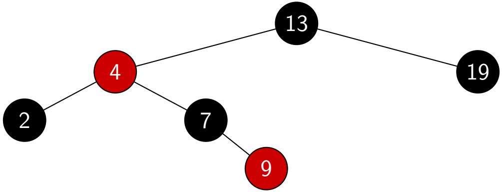
Red-Black Trees

Definition (Red-Black Tree)

A red-black tree is a binary search tree that satisfies the following red-black properties:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (`None` node) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

Quiz I: Is this a Red-Black Tree?



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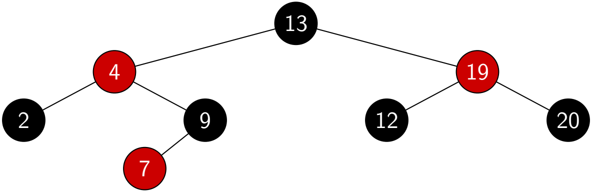

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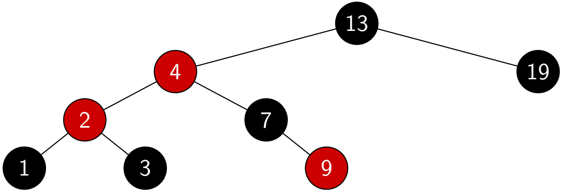

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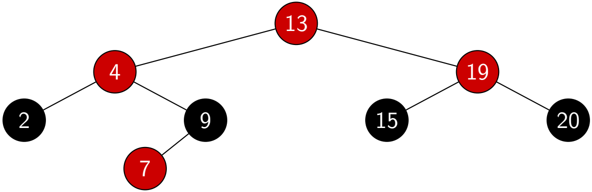

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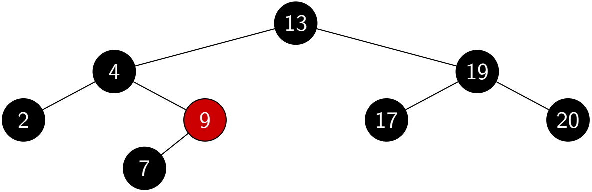

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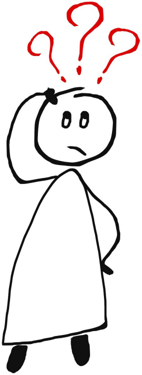
Questions



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Height of Red-Black Tree

Theorem

A red-black tree with n inner nodes has height at most $2\log_2(n+1)$.

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Let the black-height $\mathrm{bh}(x)$ of node x denote the number of black nodes on any simple path from, but not including, x down to a leaf.

We first show by induction on the height of x that the subtree rooted at any node x contains at least $2^{\mathrm{bh}(x)} - 1$ inner nodes. ...

Height of Red-Black Tree

Proof (continued).

Height of x is 0: x is a leaf and the subtree rooted at x contains $2^{\mathrm{bh}(x)} - 1 = 2^0 - 1 = 0$ inner nodes.

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Inductive step: x has positive height.
Then x has two children. If a child is black, it contributes 1 to x's black-height but not to its own. If a child is red, then it contributes to neither x's black-height nor its own.

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Therefore, each child has a black-height of $\mathrm{bh}(x) - 1$ or $\mathrm{bh}(x)$.

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Since the height of the child is smaller than the one of x, by the inductive hypothesis the subtree rooted by each child has at least $2^{\mathrm{bh}(x)-1} - 1$ inner nodes.

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Since the height of the child is smaller than the one of x, by the inductive hypothesis the subtree rooted by each child has at least $2^{\mathrm{bh}(x)-1} - 1$ inner nodes.
Thus, the subtree rooted by x contains at least $2(2^{\mathrm{bh}(x)-1} - 1) + 1 = 2^{\mathrm{bh}(x)} - 1$ inner nodes. ...

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Let h be the height of the tree. Since both children of a red node must be black, at least half of the nodes on any simple path from the root to a leaf (not including the root) must be black.
Thus, the black-height of the root is at least $h/2$ and thus $n > 2^{h/2} - 1$.

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Moving the 1 to the left-hand side and taking logarithms on both sides yields $\log_2(n+1) \geq h/2$, or $h \leq 2\log_2(n+1)$. □

Height of Red-Black Tree: Consequence

Theorem

A red-black tree with n inner nodes has height at most $2\log_2(n+1)$.

- The height of a red-black tree is in $O(\log_2 n)$.
- Red-black trees are binary search trees.
- On binary search trees, `search(n, k)`, `minimum(n)`, `maximum(n)`, `successor(n)`,`predecessor(n)` can run in time $O(h)$ (cf. Ch. B5).
- We can use the same implementation for red-black trees, achieving running time $O(\log_2(n))$ for all these queries.

Insertion (and Deletion)

Modifying Red-Black Trees

We cannot simply use the insertion and deletion implementation from binary search trees (Why not?).

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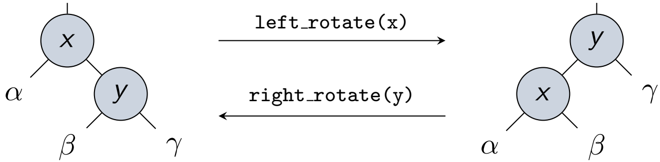
https://www.cs.usfca.edu/~galles/visualization/RedBlack.html

Insert (and delete) a number of keys into the red-black tree. What do you observe?

Rotation

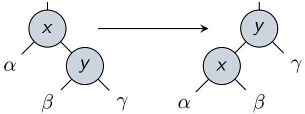
- Inserting and deleting nodes as in binary search trees does not preserve the red-black property.
- Rotation is an operation that transforms the structure of the tree but preserves the binary-search-tree property.
- Two variants: left and right rotation.
- We use them to re-establish the red-black property during an insertion/deletion.



Left-Rotation

```
class RedBlackTree:
    def __init__(self):
        self.nil = Node(None, None, color=BLACK) # sentinel
        self.root = self.nil

    def left_rotate(self, x):
        y = x.right
        x.right = y.left
        if y.left is not self.nil:
            y.left.parent = x
        y.parent = x.parent
        if x.parent is self.nil: # x was root node
            self.root = y
        elif x is x.parent.left:
            x.parent.left = y
        else:
            x.parent.right = y
        y.left = x
        x.parent = y
```



Insertion

```python
def insert(self, key, value):
    current = self.root
    parent = self.nil
    while current is not self.nil:
        parent = current
        if current.key > key:
            current = current.left
        else:
            current = current.right
    node = Node(key, value, color=RED)
    node.parent = parent
    if parent is self.nil: # tree was empty
        self.root = node
    elif key < parent.key:
        parent.left = node
    else:
        parent.right = node
    node.left = self.nil # explicit leaf nodes
    node.right = self.nil
    self.fixup(node)
```

Up to this point pretty much like insert in binary search tree.

What red-black properties can be violated before the fixup?

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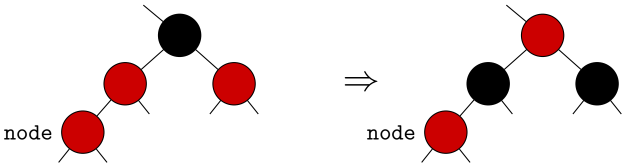
Property 2 is easy to re-establish: Just color the root black.
For property 4, distinguish three cases...

Fixup: Case 1

Potential problem: `node` and its parent are both red (the only violation of red-black property 4).

Case 1: The uncle (parent's sibling) of `node` is red.

- The grandparent of `node` cannot be red (by property 4).
- Idea: Make grandparent red and parent and uncle black.
- Afterwards: Need to fixup grandparent (its parent could be red).

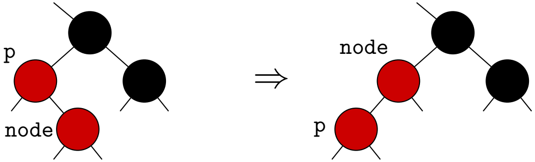


Fixup: Case 2

[Suppose `node`'s parent is a left child.]

Case 2: The uncle of `node` is black and `node` is a right child.

- Perform a left-rotation on the parent.
- Now the red previous parent is the left child of the red node.
- This constellation corresponds to case 3 (with the previous parent in the role of the red child node) and is resolved the same way (next slide).

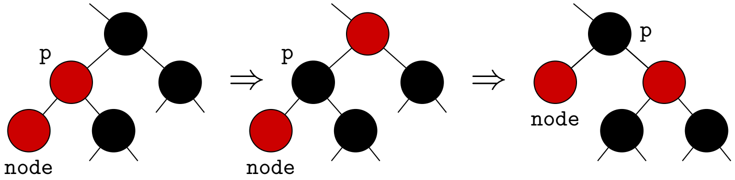


Fixup: Case 3

[Suppose `node`'s parent is a left child.]

Case 3: The uncle of `node` is black and `node` is a left child.

- Make parent black and grandparent red.
- Afterwards, perform a right-rotation on the grandparent.



Insertion: Fixup

```python
def fixup(self, node):
    while node.parent.color == RED:
        grandparent = node.parent.parent
        if node.parent is grandparent.left:
            uncle = grandparent.right
            if uncle.color == RED:
                node.parent.color = BLACK
                uncle.color = BLACK
                grandparent.color = RED
                node = grandparent
            else:
                if node is node.parent.right:
                    node = node.parent
                    self.left_rotate(node)
                node.parent.color = BLACK
                node.parent.parent.color = RED
                self.right_rotate(grandparent)
        else:
            ...
            # symmetric cases 1-3, where parent is
            # not the left child (cf. notebook).
    self.root.color = BLACK
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Case 1

Case 2

Case 3

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Case 1 (lines `node.parent.color = BLACK` … `node = grandparent`)

Case 2 (lines `node = node.parent`, `self.left_rotate(node)`)

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Running time?

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Running time: $O(h)$

(h tree height)

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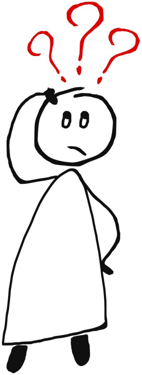

Questions?

Deletion

- Deleting a node from a red-black tree is more complicated than inserting a node.
- We do not cover the details in this course.
- Deletion from a tree with n nodes is possible in time $O(\log_2 n)$.

Summary

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- Red-black trees are a special kind of binary search trees that are approximately balanced.
- The height of a red-black tree with n nodes is $O(\log_2 n)$.
- Consequently, the query operations only take logarithmic time in the size of the tree.
- The same is true for insertion and deletion.