Algorithms and Data Structures

B6. Red-Black Trees

Gabriele Röger

University of Basel

April 18/24, 2024

Algorithms and Data Structures

April 18/24, 2024 — B6. Red-Black Trees

B6.1 Red-Black Trees

Motivation

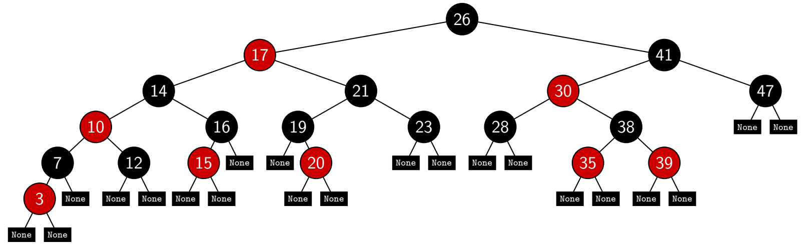
- Binary search trees can support many relevant operations in linear time in the height of the tree.
- But: Binary search trees can degenerate into chains, in which case the operations take linear time in the number n of elements (no better than with a linked list).
- Idea: Search-trees schemes that are in some form "balanced" and can guarantee running time $O(\log_2 n)$ in the worst case.
  - AVL trees: for every node, the height of the left and right subtree differs by at most 1.
  - B-trees: permit several keys and subtrees per node (e.g. special case: 2-3 tree).
  - Red-Black trees: use node colors to maintain an approximate balancing.
  - ...

Red-Black Trees: Representation

- Use one extra bit per node, storing its color, which can be either red or black.
- Each node now contains attributes color, key, left, right and parent.

None Leaf Nodes

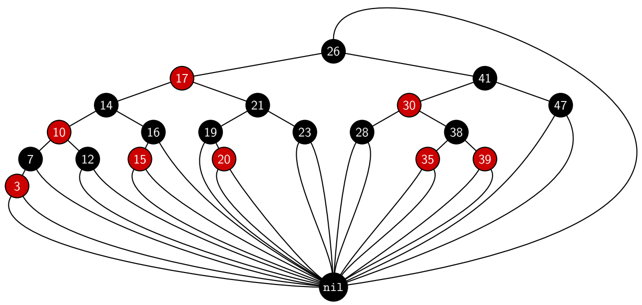
- Left, right and parent are `None` if there is no corresponding node.
- Because it is conceptionally and implementation-wise easier, we will represent them as actual node objects.
- These are then the leaves of the trees and the nodes holding the entries are inner nodes.



None Leaf Nodes: Sentinel

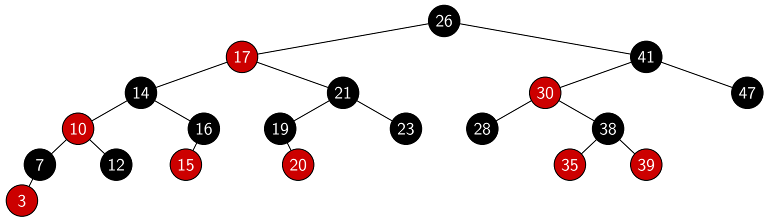
Instead of many leaf nodes, we use a single sentinel node `nil`.

- Implemented like a normal (black) node but used as child of many nodes.
- The sentinel also serves as parent of the root.
- Attributes for parent and children can take on arbitrary values.



Graphical Representation

On the slides, we omit the `None` leaf nodes/sentinel:



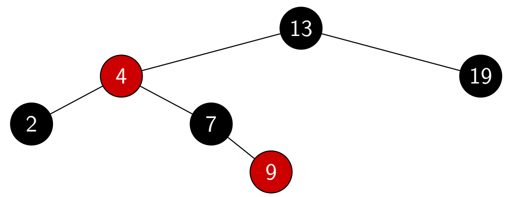
Red-Black Trees

Definition (Red-Black Tree)

A red-black tree is a binary search tree that satisfies the following red-black properties:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (`None` node) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

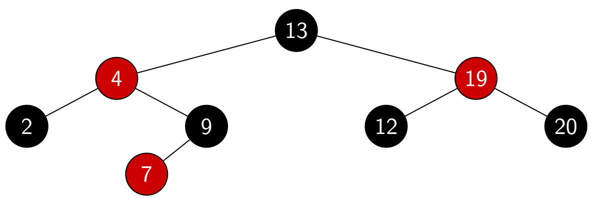
Quiz I: Is this a Red-Black Tree?



Reminder: A red-black tree is a binary search tree where:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (`None` node) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

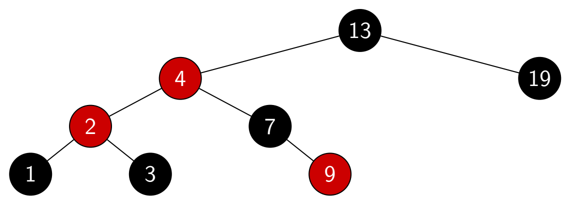
Quiz II: Is this a Red-Black Tree?



Reminder: A red-black tree is a binary search tree where:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (`None` node) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

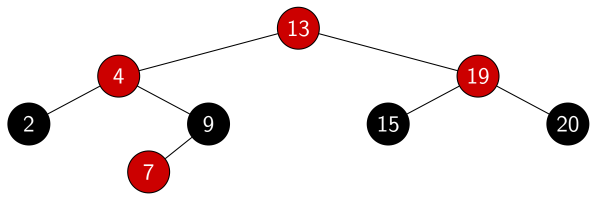
Quiz III: Is this a Red-Black Tree?



Reminder: A red-black tree is a binary search tree where:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (`None` node) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

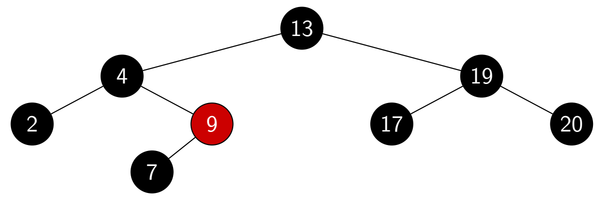
Quiz IV: Is this a Red-Black Tree?



Reminder: A red-black tree is a binary search tree where:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (`None` node) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

Quiz V: Is this a Red-Black Tree?



Reminder: A red-black tree is a binary search tree where:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (`None` node) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

Height of Red-Black Tree

Theorem
A red-black tree with n inner nodes has height at most $2\log_2(n+1)$.

Proof
Let the black-height $\mathrm{bh}(x)$ of node x denote the number of black nodes on any simple path from, but not including, x down to a leaf.

We first show by induction on the height of x that the subtree rooted at any node x contains at least $2^{\mathrm{bh}(x)} - 1$ inner nodes. ...

Height of Red-Black Tree

Proof (continued).
Height of x is 0: x is a leaf and the subtree rooted at x contains $2^{\mathrm{bh}(x)} - 1 = 2^0 - 1 = 0$ inner nodes.

Inductive step: x has positive height.
Then x has two children. If a child is black, it contributes 1 to x's black-height but not to its own. If a child is red, then it contributes to neither x's black-height nor its own.
Therefore, each child has a black-height of $\mathrm{bh}(x) - 1$ or $\mathrm{bh}(x)$.
Since the height of the child is smaller than the one of x, by the inductive hypothesis the subtree rooted by each child has at least $2^{\mathrm{bh}(x)-1} - 1$ inner nodes.
Thus, the subtree rooted by x contains at least $2(2^{\mathrm{bh}(x)-1} - 1) + 1 = 2^{\mathrm{bh}(x)} - 1$ inner nodes. ...

Height of Red-Black Tree

Proof (continued).

We showed that that the subtree rooted at any node x contains at least $2^{\mathrm{bh}(x)} - 1$ inner nodes.

Let h be the height of the tree. Since both children of a red node must be black, at least half of the nodes on any simple path from the root to a leaf (not including the root) must be black.
Thus, the black-height of the root is at least $h/2$ and thus $n > 2^{h/2} - 1$.

Moving the 1 to the left-hand side and taking logarithms on both sides yields $\log_2(n+1) \geq h/2$, or $h \leq 2\log_2(n+1)$. □

Height of Red-Black Tree: Consequence

Theorem

A red-black tree with n inner nodes has height at most $2\log_2(n+1)$.

- The height of a red-black tree is in $O(\log_2 n)$.
- Red-black trees are binary search trees.
- On binary search trees, `search(n, k)`, `minimum(n)`, `maximum(n)`, `successor(n)`,`predecessor(n)` can run in time $O(h)$ (cf. Ch. B5).
- We can use the same implementation for red-black trees, achieving running time $O(\log_2(n))$ for all these queries.

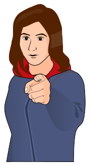
B6.2 Insertion (and Deletion)

Modifying Red-Black Trees

We cannot simply use the insertion and deletion implementation from binary search trees (Why not?).

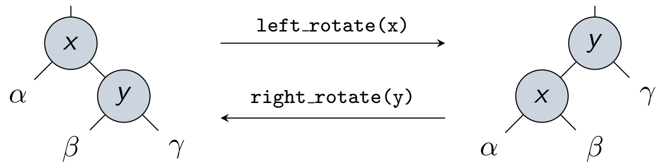
`https://www.cs.usfca.edu/~galles/visualization/RedBlack.html`

Insert (and delete) a number of keys into the red-black tree. What do you observe?



Rotation

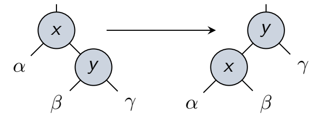
- Inserting and deleting nodes as in binary search trees does not preserve the red-black property.
- Rotation is an operation that transforms the structure of the tree but preserves the binary-search-tree property.
- Two variants: left and right rotation.
- We use them to re-establish the red-black property during an insertion/deletion.



Left-Rotation

```
class RedBlackTree:
    def __init__(self):
        self.nil = Node(None, None, color=BLACK) # sentinel
        self.root = self.nil

    def left_rotate(self, x):
        y = x.right
        x.right = y.left
        if y.left is not self.nil:
            y.left.parent = x
        y.parent = x.parent
        if x.parent is self.nil: # x was root node
            self.root = y
        elif x is x.parent.left:
            x.parent.left = y
        else:
            x.parent.right = y
        y.left = x
        x.parent = y
```



Insertion

```
def insert(self, key, value):
    current = self.root
    parent = self.nil
    while current is not self.nil:
        parent = current
        if current.key > key:
            current = current.left
        else:
            current = current.right
    node = Node(key, value, color=RED)
    node.parent = parent
    if parent is self.nil: # tree was empty
        self.root = node
    elif key < parent.key:
        parent.left = node
    else:
        parent.right = node
    node.left = self.nil # explicit leaf nodes
    node.right = self.nil
    self.fixup(node)
```

Up to this point pretty much like insert in binary search tree.

What red-black properties can be violated before the fixup?

Reminder: Red-Black Trees

Definition (Red-Black Tree)

A red-black tree is a binary search tree that satisfies the following red-black properties:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (None node) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

What could be violated before the fixup? Only 2 or 4!

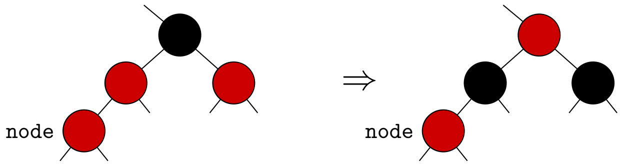
Property 2 is easy to re-establish: Just color the root black.
For property 4, distinguish three cases...

Fixup: Case 1

Potential problem: `node` and its parent are both red
(the only violation of red-black property 4).

Case 1: The uncle (parent's sibling) of `node` is red.

- The grandparent of `node` cannot be red (by property 4).
- Idea: Make grandparent red and parent and uncle black.
- Afterwards: Need to fixup grandparent (its parent could be red).

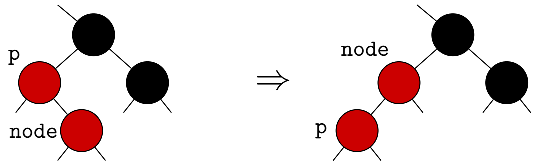


Fixup: Case 2

[Suppose `node`'s parent is a left child.]

Case 2: The uncle of `node` is black and `node` is a right child.

- Perform a left-rotation on the parent.
- Now the red previous parent is the left child of the red node.
- This constellation corresponds to case 3 (with the previous parent in the role of the red child node) and is resolved the same way (next slide).

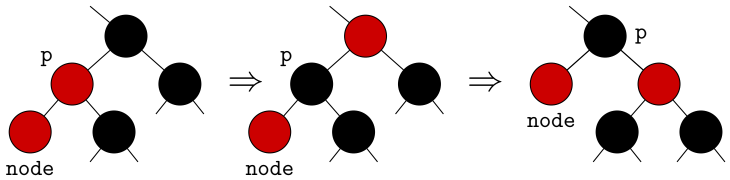


Fixup: Case 3

[Suppose `node`'s parent is a left child.]

Case 3: The uncle of `node` is black and `node` is a left child.

- Make parent black and grandparent red.
- Afterwards, perform a right-rotation on the grandparent.



Insertion: Fixup

```python
def fixup(self, node):
    while node.parent.color == RED:
        grandparent = node.parent.parent
        if node.parent is grandparent.left:
            uncle = grandparent.right
            if uncle.color == RED:
                node.parent.color = BLACK
                uncle.color = BLACK
                grandparent.color = RED
                node = grandparent
            else:
                if node is node.parent.right:
                    node = node.parent
                    self.left_rotate(node)
                node.parent.color = BLACK
                node.parent.parent.color = RED
                self.right_rotate(grandparent)
        else:
            ...
            # symmetric cases 1-3, where parent is
            # not the left child (cf. notebook).
    self.root.color = BLACK
```

Case 1

Case 2

Case 3

Running time: $O(h)$
(h tree height)

Insertion: Running Time

```python
def insert(self, key, value):
    current = self.root
    parent = self.nil
    while current is not self.nil:
        parent = current
        if current.key > key:
            current = current.left
        else:
            current = current.right
    node = Node(key, value, color=RED)
    node.parent = parent
    if parent is self.nil: # tree was empty
        self.root = node
    elif key < parent.key:
        parent.left = node
    else:
        parent.right = node
    node.left = self.nil # explicit leaf nodes
    node.right = self.nil
    self.fixup(node)
```

Running time:
$O(h)$
(h tree height)

Deletion

- Deleting a node from a red-black tree is more complicated than inserting a node.
- We do not cover the details in this course.
- Deletion from a tree with n nodes is possible in time $O(\log_2 n)$.

B6.3 Summary

Summary

- Red-black trees are a special kind of binary search trees that are approximately balanced.
- The height of a red-black tree with n nodes is $O(\log_2 n)$.
- Consequently, the query operations only take logarithmic time in the size of the tree.
- The same is true for insertion and deletion.