Algorithms and Data Structures

B1. Arrays and Linked Lists

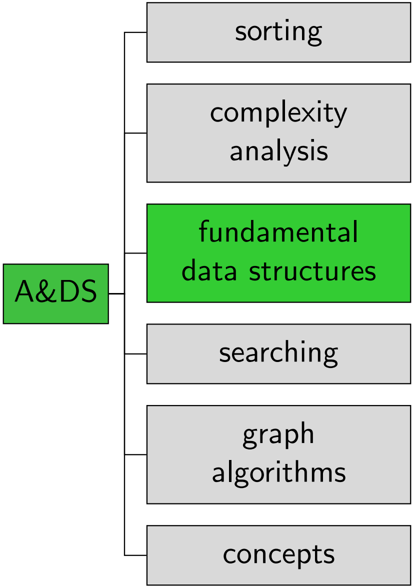
Gabriele Röger

University of Basel

March 27/April 3, 2024

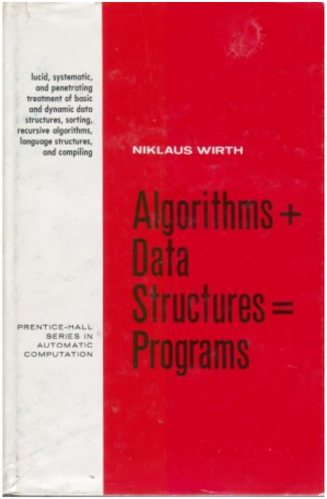
Data Structures

Content of the Course

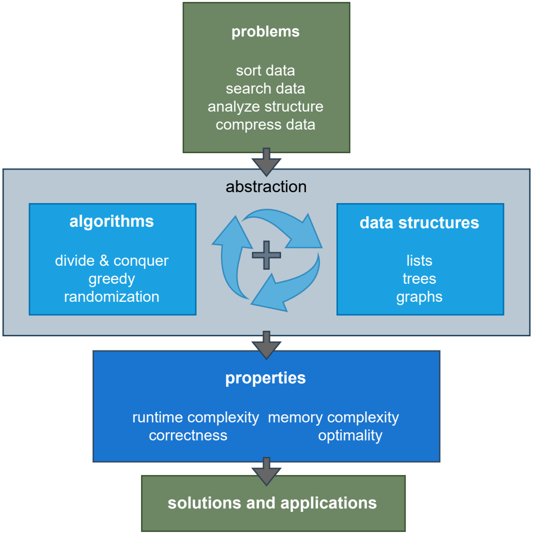


Data Structures

- Programming goes beyond writing algorithms.
  - Organisation of data is central.
- Elegant data structures lead to elegant code.
- Programmers. . .
  - need a catalogue of data structures, and
  - need to know their characteristics.


lucid, systematic, and penetrating treatment of basic and dynamic data structures, sorting, recursive algorithms, language structures, and compiling
PRENTICE-HALL SERIES IN AUTOMATIC COMPUTATION
NIKLAUS WIRTH
Algorithms + Data Structures = Programs

Overview


problems
sort data
search data
analyze structure
compress data
abstraction
algorithms
divide & conquer
greedy
randomization
data structures
lists
trees
graphs
properties
runtime complexity
memory complexity
correctness
optimality
solutions and applications

Data Structures

Bad programmers worry about the code. Good programmers worry about data structures and their relationships.

Linus Torwalds

Data Structures

Show me your algorithm and conceal your data structures,
and I shall continue to be mystified.

Show me your data structures, and I won't usually need your algorithm; it will be obvious.

Fred Brooks (paraphrased)

Arrays

Data Structure: Array

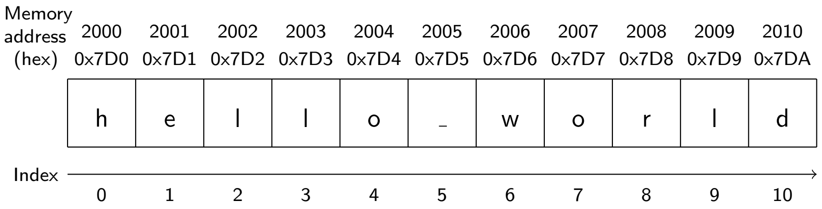
- Arrays are one of the fundamental data structures, that can be found in (almost) every programming language.
- An array stores a sequence of elements (of the same memory size) as a contiguous sequence of bytes in memory.
- The number of elements is fixed.
- We can access elements by their index.

In Java:

```
byte[] myByteArray = new byte[100];
char[] myCharArray = new char[50];
```

Example: char Array

- One char occupies 1 byte.
- The first element is at memory address 2000 (7D0 in hexadecimal).
- The first element has index 0.
- Then the element with index i is at address $2000 + \mathrm{i}$.

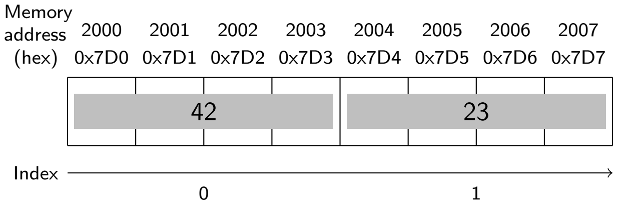


Array: Position of i-th Element Easy to Compute

In general:

- First position typically indexed with 0 or 1.
  In the following, s for the index of the first element.
- Suppose the array starts at memory address a and each array element occupies b bytes.
- Then the element with index i occupies bytes $a + b(i - s)$ to $a + b(i - s + 1) - 1$.

With 32-bit integers (4 byte)



Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
    - the size of the array (number of elements) and
    - the address of the start of the allocated memory.

Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
  - the size of the array (number of elements) and
  - the address of the start of the allocated memory.
- What is the running time of the following operations (relative to the size *n* of the array)?
  - `get(i)` – return element at position `i`
  - `set(i, x)` – write object `x` to position `i`
  - `length()` – return length of the array
  - `find(x)` – return index of element `x` or None if not included.

Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
  - the size of the array (number of elements) and
  - the address of the start of the allocated memory.
- What is the running time of the following operations (relative to the size *n* of the array)?
  - `get(i)` – return element at position `i` $\rightsquigarrow \Theta(1)$
  - `set(i, x)` – write object `x` to position `i`
  - `length()` – return length of the array
  - `find(x)` – return index of element `x` or None if not included.

Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
    - the size of the array (number of elements) and
    - the address of the start of the allocated memory.
- What is the running time of the following operations (relative to the size n of the array)?
    - `get(i)` – return element at position `i` $\rightsquigarrow \Theta(1)$
    - `set(i, x)` – write object `x` to position `i` $\rightsquigarrow \Theta(1)$
    - `length()` – return length of the array
    - `find(x)` – return index of element `x` or None if not included.

Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
  - the size of the array (number of elements) and
  - the address of the start of the allocated memory.
- What is the running time of the following operations (relative to the size n of the array)?
  - `get(i)` – return element at position `i` $\rightsquigarrow \Theta(1)$
  - `set(i, x)` – write object `x` to position `i` $\rightsquigarrow \Theta(1)$
  - `length()` – return length of the array $\rightsquigarrow \Theta(1)$
  - `find(x)` – return index of element `x` or None if not included.

Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
  - the size of the array (number of elements) and
  - the address of the start of the allocated memory.
- What is the running time of the following operations (relative to the size n of the array)?
  - `get(i)` – return element at position `i` $\rightsquigarrow \Theta(1)$
  - `set(i, x)` – write object `x` to position `i` $\rightsquigarrow \Theta(1)$
  - `length()` – return length of the array $\rightsquigarrow \Theta(1)$
  - `find(x)` – return index of element `x` or None if not included.

    $\rightsquigarrow$ iterates over the array and stops if element found.

    $\rightsquigarrow$ Best case $\Theta(1)$, Avg. and worst case $\Theta(n)$

Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
  - the size of the array (number of elements) and
  - the address of the start of the allocated memory.
- What is the running time of the following operations (relative to the size n of the array)?
  - `get(i)` – return element at position `i` $\rightsquigarrow \Theta(1)$
  - `set(i, x)` – write object `x` to position `i` $\rightsquigarrow \Theta(1)$
  - `length()` – return length of the array $\rightsquigarrow \Theta(1)$
  - `find(x)` – return index of element `x` or None if not included.
    $\rightsquigarrow$ iterates over the array and stops if element found.
    $\rightsquigarrow$ Best case $\Theta(1)$, Avg. and worst case $\Theta(n)$
- What is the memory complexity?

Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
  - the size of the array (number of elements) and
  - the address of the start of the allocated memory.
- What is the running time of the following operations (relative to the size n of the array)?
  - `get(i)` – return element at position `i` $\rightsquigarrow \Theta(1)$
  - `set(i, x)` – write object `x` to position `i` $\rightsquigarrow \Theta(1)$
  - `length()` – return length of the array $\rightsquigarrow \Theta(1)$
  - `find(x)` – return index of element `x` or None if not included.
    $\rightsquigarrow$ iterates over the array and stops if element found.
    $\rightsquigarrow$ Best case $\Theta(1)$, Avg. and worst case $\Theta(n)$
- What is the memory complexity?

Observation

Complexity is direct consequence of data representation.

Lists in Python

- Python lists can contain arbitrarily mixed objects.
  e.g. `["word", 42, ([39, "hi"])]`

Lists in Python

- Python lists can contain arbitrarily mixed objects.
  e.g. `["word", 42, ([39, "hi"])]`
  - Elements “live” somewhere else in memory.
  - The memory range of the array only stores their address.

Lists in Python

- Python lists can contain arbitrarily mixed objects.
  e.g. `["word", 42, ([39, "hi"])]`
  - Elements “live” somewhere else in memory.
  - The memory range of the array only stores their address.
- Python lists do not have a fixed size.
  e.g. `["word", 42, ([39, "hi"])].append(3)`

Lists in Python

- Python lists can contain arbitrarily mixed objects.
  e.g. `["word", 42, ([39, "hi"])]`
  - Elements "live" somewhere else in memory.
  - The memory range of the array only stores their address.
- Python lists do not have a fixed size.
  e.g. `["word", 42, ([39, "hi"])].append(3)`
  $\rightarrow$ dynamic array

Dynamic Arrays

(Static) arrays have fixed capacity that must be specified at allocation.

- Need arrays that can grow dynamically.
- Runtime complexity of previous operations should be preserved.

Dynamic Arrays

(Static) arrays have fixed capacity that must be specified at allocation.

- Need arrays that can grow dynamically.
- Runtime complexity of previous operations should be preserved.

Additional operations:

- `append(x)` (or `push`) – append element `x` at the end.
- `insert(i, x)` – insert element `x` at position `i`.
- `pop()` - remove the last element.
- `remove(i)` - remove the element at position `i`.

Changing the Array Size: Naive Method

- `append` (and `insert`) increase the size of the array.
- `pop` decreases the size.
- Naive method:
  - Allocate new memory range that is one element larger/smaller.
  - Move all (but the potentially popped) element over.

Changing the Array Size: Naive Method

- `append` (and `insert`) increase the size of the array.
- `pop` decreases the size.
- Naive method:
  - Allocate new memory range that is one element larger/smaller.
  - Move all (but the potentially popped) element over.

With this approach, these operations would take linear time in the current size of the array!

Better Approach: Overallocate Memory

- Allocate more memory than needed for the current array size.
- Distinguish
  - capacity $=$ number of elements that fit in the allocated space.
  - size $=$ number of currently contained elements.

Better Approach: Append/Insert

Append

- If capacity $>$ size:
    - Write the new element to position size and increment size.
- Otherwise (capacity $=$ size):
    - Allocate new memory that is larger than necessary (e.g. twice the previous capacity).
    - Copy all elements to the new memory (release the old one).
    - Update the capacity and continue as in case capacity $>$ size.

Insert at pos i: Analogously but move all elements at positions i to size-1 one position to the right before writing the new element to i.

Better Approach: Pop/Remove

- If capacity much too large (e.g. capacity $> 4 \cdot$ size), move all elements into new smaller memory range (e.g. with half the previous capacity)
- Pop: remove element at position size - 1 and decrement size.
- Remove: remove element at position i and move all elements right of i one position to the left, decrepement size.

Amortized Analysis

- Worst-case analysis often pessimistic: append takes linear time if new memory allocated but in a sequence of append operations, this will happen rarely.
- Amortized analysis determines the average cost of an operation over an entire sequence of operations.
- Don't confuse this with an average-case analysis.
- Different methods
  - Aggregate analysis
  - Accounting method $\leftarrow$ now
  - Potential method

Accounting Method

- Assign charges to operations.
- Some operations charged more or less than they actually cost.
- If charged more: save difference as credit
- If charged less: use up some credit to pay for the difference.
- Credit must be non-negative all the time.
- Then the total amortized cost is always an upper bound on the actual total costs so far.

Accounting Method: Append I

- Append without resize: constant cost (e.g. 1).
  Just insert the element at the right position.
- Append with resize: linear cost (1 for every element).
  - If the append element gets position 2^i ($i \in \mathbb{N}_{>0}$),
  - we first allocate overall space for 2^{i+1} elements, and
  - move all $2^i - 1$ existing elements to the new space.

Accounting Method: Append I

- Append without resize: constant cost (e.g. 1).
  Just insert the element at the right position.
- Append with resize: linear cost (1 for every element).
  - If the append element gets position 2^i ($i \in \mathbb{N}_{>0}$),
  - we first allocate overall space for 2^{i+1} elements, and
  - move all $2^i - 1$ existing elements to the new space.
- Starting from an empty array executing a sequence of append operations, we observe cost sequence
  $1, 1, 3, 1, 5, 1, 1, 1, 9, 1, 1, 1, 1, 1, 1, 1, 17, 1 \ldots$

Accounting Method: Append II

Charge cost 3 for every append operation.

size (after append)	capacity	charge	cost	credit
1	2	3	1	2
2	2	3	1	4
3	4	3	3	4
4	4	3	1	6
5	8	3	5	4
6	8	3	1	6
7	8	3	1	8
8	8	3	1	10
9	16	3	9	4
10	16	3	1	6

Charging 3 per operation covers all “running time costs”.
$\rightarrow$ Append has constant amortized running time.

Worst-Case Running Time Array

Operation	Array
Access element by position	$O(1)$
Prepend/remove first element	$O(n)$
Append	$O(1)$ (amortized)
Remove last element	$O(1)$ (amortized)
Insert, remove from the middle	$O(n)$
Traverse all elements	$O(n)$

Linked Lists

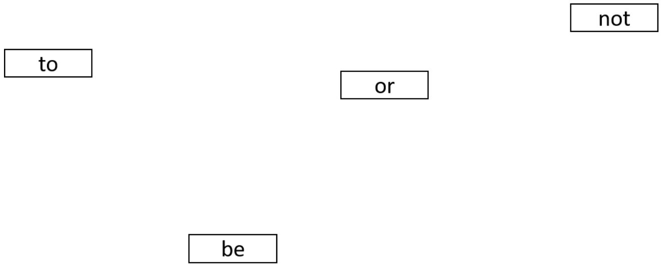
Motivation

- Arrays need a large continuous block of memory.
- Inserting elements at arbitrary positions is expensive.

Alternative that allows us to distribute the elements in memory?

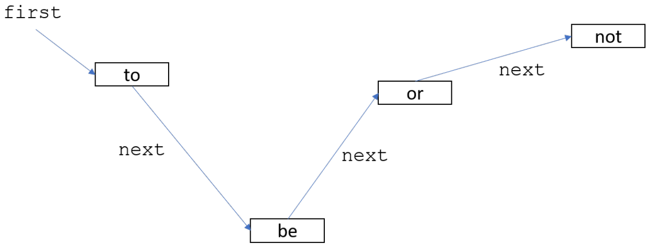
Question?

- How can we order elements that are distributed in memory?



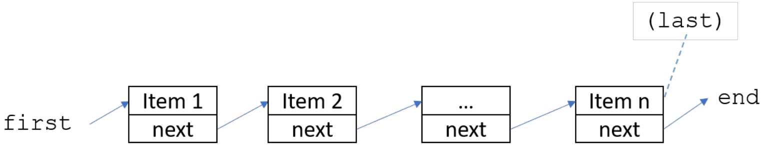
Question?

- How can we order elements that are distributed in memory?



Linked Lists

- Every node stores its entry as well as a reference (pointer) to its successor.
- Need special value for the next pointer of the last element.
- . . . or a reference to the last element.



Jupyter Notebook

Jupyter notebook: `linked_lists.ipynb`

Implementation: Node

```
class Node:
    def __init__(self, item, next=None):
        self.item = item
        self.next = next
```

Implementation: List (without last reference)

```python
class LinkedList:
    def __init__(self):
        self.first = None

    # prepend item at the front of the list
    def prepend(self, item):
        new_node = Node(item, self.first)
        self.first = new_node

    ... # other methods added to notebook after lecture
```

Worst-Case Running Time Array / Linked List

Operation	Array	Linked List
Prepend/remove first element	$O(n)$	$O(1)$
Append	$O(1)$ (amortized)	$O(n)$
Remove last element	$O(1)$ (amortized)	$O(n)$
Insert, remove from the middle	$O(n)$	$O(n)$
Traverse all elements	$O(n)$	$O(n)$
Find an element	$O(n)$	$O(n)$
Access element by position	$O(1)$	–

What running times could we improve if we also maintained a pointer to the last element of the linked list?

Worst-Case Running Time Array / Linked List

Operation	Array	Linked List
Prepend/remove first element	$O(n)$	$O(1)$
Append	$O(1)$ (amortized)	$O(n)$
Remove last element	$O(1)$ (amortized)	$O(n)$
Insert, remove from the middle	$O(n)$	$O(n)$
Traverse all elements	$O(n)$	$O(n)$
Find an element	$O(n)$	$O(n)$
Access element by position	$O(1)$	–

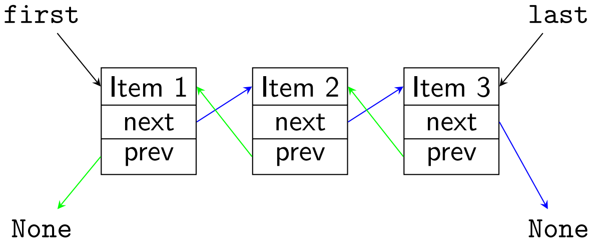
What running times could we improve if we also maintained a pointer to the last element of the linked list?

Take-home Message

- Different data structures have different trade-offs.

Doubly Linked Lists

- Idea: Do not only store a reference to the successor but also to the predecessor.
- Renders appending at/removal from end constant time.



Jupyter Notebook

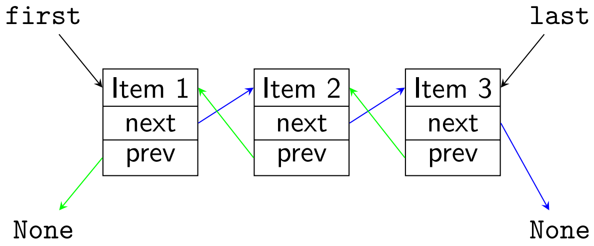
Jupyter notebook: `doubly_linked_lists.ipynb`

Doubly Linked Lists: Implementation

```python
class Node:
    def __init__(self, item, next=None, prev=None):
        self.item = item
        self.next = next
        self.prev = prev

class DoublyLinkedList:
    def __init__(self):
        self.first = None
        self.last = None

    def is_empty(self):
        return self.first is None

    # other methods on next slides
```

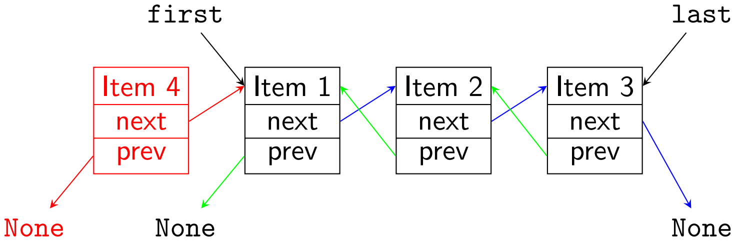
Doubly Linked Lists: prepend

```
    def prepend(self, item):
        if self.is_empty():
            self.first = Node(item)
            self.last = self.first
        else:
            node = Node(item, self.first, None)
            self.first.prev = node
            self.first = node
```



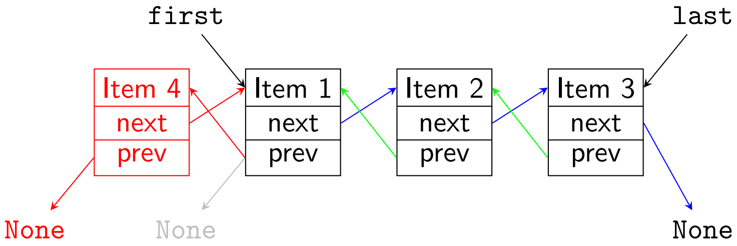
Doubly Linked Lists: prepend

```
    def prepend(self, item):
        if self.is_empty():
            self.first = Node(item)
            self.last = self.first
        else:
            node = Node(item, self.first, None)
            self.first.prev = node
            self.first = node
```



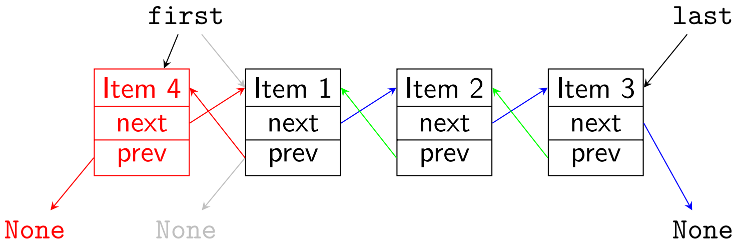
Doubly Linked Lists: prepend

```
    def prepend(self, item):
        if self.is_empty():
            self.first = Node(item)
            self.last = self.first
        else:
            node = Node(item, self.first, None)
            self.first.prev = node
            self.first = node
```



Doubly Linked Lists: prepend

```python
    def prepend(self, item):
        if self.is_empty():
            self.first = Node(item)
            self.last = self.first
        else:
            node = Node(item, self.first, None)
            self.first.prev = node
            self.first = node
```



Doubly Linked Lists: append

```
    def append(self, item):
        if self.is_empty():
            self.first = Node(item)
            self.last = self.first
        else:
            node = Node(item, None, self.last)
            self.last.next = node
            self.last = node
```

Doubly Linked Lists: remove_first

```
    def remove_first(self):
        if self.is_empty():
            raise Exception("removing from empty list")
        item = self.first.item
        self.first = self.first.next
        if self.first is not None:
            self.first.prev = None
        else:
            self.last = None
        return item
```

Doubly Linked Lists: remove_last

With doubly linked lists, removing the last element is analogous to removing the first element:

```
    def remove_last(self):
        if self.is_empty():
            raise Exception("removing from empty list")
        item = self.last.item
        self.last = self.last.prev
        if self.last is not None:
            self.last.next = None
        else:
            self.first = None
        return item
```

Worst-Case Running Time Array / Doubly Linked List

Operation	Array	Doubly Linked List
Prepend/remove first element	$O(n)$	$O(1)$
Append	$O(1)$ (amort.)	$O(1)$
Remove last element	$O(1)$ (amort.)	$O(1)$
Insert, remove in the middle	$O(n)$	$O(n)/O(1)^*$
Traverse all elements	$O(n)$	$O(n)$
Find an element	$O(n)$	$O(n)$
Access element by position	$O(1)$	–

* constant, if node at the position is parameter

Worst-Case Running Time Array / Doubly Linked List

Operation	Array	Doubly Linked List
Prepend/remove first element	$O(n)$	$O(1)$
Append	$O(1)$ (amort.)	$O(1)$
Remove last element	$O(1)$ (amort.)	$O(1)$
Insert, remove in the middle	$O(n)$	$O(n)/O(1)^*$
Traverse all elements	$O(n)$	$O(n)$
Find an element	$O(n)$	$O(n)$
Access element by position	$O(1)$	–

* constant, if node at the position is parameter

Take-home Message

Compared to singly linked lists, doubly linked lists need a linear amount of additional memory (for the `prev` references), but provide better running times for operations at the end of the list.

Summary

Summary

- An amortized analysis determines the average cost of an operation over an entire sequence of operations.
- Arrays and linked lists store sequences of items.
  - Arrays store items in a continuous space and can efficiently access an item by index.
  - Linked lists store items in nodes with a reference to the next node (doubly linked lists: also to previous node).