# Algorithms and Data Structures

## B1. Arrays and Linked Lists

Gabriele Röger

University of Basel

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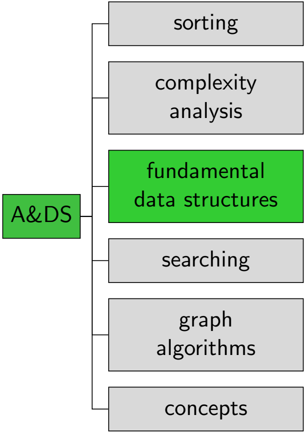
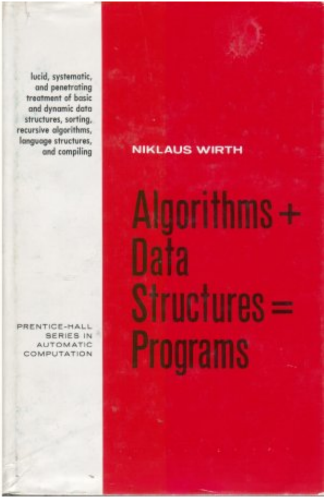
# B1.1 Data Structures

# Content of the Course

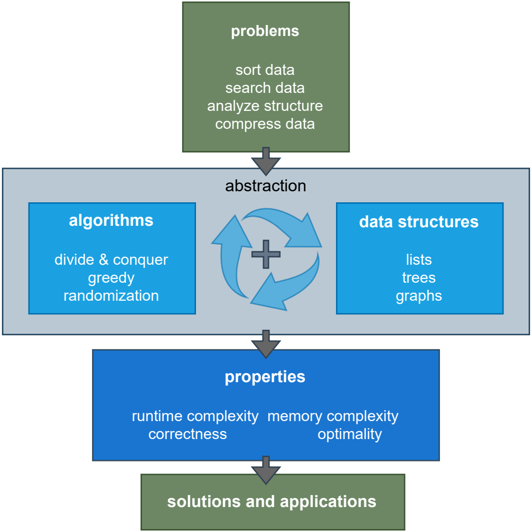


# Data Structures

- Programming goes beyond writing algorithms.
  - Organisation of data is central.
- Elegant data structures lead to elegant code.
- Programmers. . .
  - need a catalogue of data structures, and
  - need to know their characteristics.



# Overview


problems
sort data
search data
analyze structure
compress data
abstraction
algorithms
divide & conquer
greedy
randomization
data structures
lists
trees
graphs
properties
runtime complexity
correctness
memory complexity
optimality
solutions and applications

# Data Structures

> Bad programmers worry about the code. Good programmers worry about data structures and their relationships.
>
> Linus Torwalds

# Data Structures

> Show me your algorithm and conceal your data structures,
> and I shall continue to be mystified.
>
> Show me your data structures, and I won't usually need your algorithm; it will be obvious.
>
> Fred Brooks (paraphrased)

# B1.2 Arrays

# Data Structure: Array

- Arrays are one of the fundamental data structures, that can be found in (almost) every programming language.
- An array stores a sequence of elements (of the same memory size) as a contiguous sequence of bytes in memory.
- The number of elements is fixed.
- We can access elements by their index.

In Java:

```
byte[] myByteArray = new byte[100];
char[] myCharArray = new char[50];
```

# Example: char Array

- One char occupies 1 byte.
- The first element is at memory address 2000 (7D0 in hexadecimal).
- The first element has index 0.
- Then the element with index *i* is at address $2000 + \mathrm{i}$.

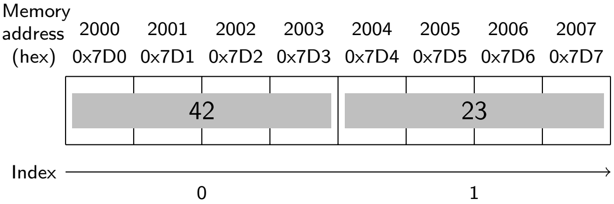
| Memory address (hex) | 2000 0x7D0 | 2001 0x7D1 | 2002 0x7D2 | 2003 0x7D3 | 2004 0x7D4 | 2005 0x7D5 | 2006 0x7D6 | 2007 0x7D7 | 2008 0x7D8 | 2009 0x7D9 | 2010 0x7DA |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | h | e | l | l | o | ␣ | w | o | r | l | d |
| Index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

# Array: Position of $i$-th Element Easy to Compute

In general:

- First position typically indexed with 0 or 1.
  In the following, $s$ for the index of the first element.
- Suppose the array starts at memory address $a$ and each array element occupies $b$ bytes.
- Then the element with index $i$ occupies bytes $a + b(i - s)$ to $a + b(i - s + 1) - 1$.

With 32-bit integers (4 byte)



# Operations and their Running Time?

- Size of entry is constant for a specific array type (such as an int array).
- After allocating the memory, the array stores
  - the size of the array (number of elements) and
  - the address of the start of the allocated memory.
- What is the running time of the following operations (relative to the size $n$ of the array)?
  - `get(i)` – return element at position `i` $\rightsquigarrow \Theta(1)$
  - `set(i, x)` – write object `x` to position `i` $\rightsquigarrow \Theta(1)$
  - `length()` – return length of the array $\rightsquigarrow \Theta(1)$
  - `find(x)` – return index of element `x` or None if not included.
    $\rightsquigarrow$ iterates over the array and stops if element found.
    $\rightsquigarrow$ Best case $\Theta(1)$, Avg. and worst case $\Theta(n)$
- What is the memory complexity?

**Observation**

Complexity is direct consequence of data representation.

# Lists in Python

- Python lists can contain arbitrarily mixed objects.
  e.g. `["word", 42, ([39, "hi"])]`
  - Elements “live” somewhere else in memory.
  - The memory range of the array only stores their address.
- Python lists do not have a fixed size.
  e.g. `["word", 42, ([39, "hi"])].append(3)`
  → dynamic array

# Dynamic Arrays

(Static) arrays have fixed capacity that must be specified at allocation.

- Need arrays that can grow dynamically.
- Runtime complexity of previous operations should be preserved.

Additional operations:

- `append(x)` (or `push`) – append element `x` at the end.
- `insert(i, x)` – insert element `x` at position `i`.
- `pop()` - remove the last element.
- `remove(i)` - remove the element at position `i`.

# Changing the Array Size: Naive Method

- `append` (and `insert`) increase the size of the array.
- `pop` decreases the size.
- Naive method:
  - Allocate new memory range that is one element larger/smaller.
  - Move all (but the potentially popped) element over.

With this approach, these operations would take linear time in the current size of the array!

# Better Approach: Overallocate Memory

- Allocate more memory than needed for the current array size.
- Distinguish
  - capacity = number of elements that fit in the allocated space.
  - size = number of currently contained elements.

# Better Approach: Append/Insert

Append

- If capacity $>$ size:
  - Write the new element to position size and increment size.
- Otherwise (capacity $=$ size):
  - Allocate new memory that is larger than necessary (e.g. twice the previous capacity).
  - Copy all elements to the new memory (release the old one).
  - Update the capacity and continue as in case capacity $>$ size.

Insert at pos $i$: Analogously but move all elements at positions $i$ to size-1 one position to the right before writing the new element to $i$.

# Better Approach: Pop/Remove

- If capacity much too large (e.g. capacity $> 4 \cdot$ size), move all elements into new smaller memory range (e.g. with half the previous capacity)
- Pop: remove element at position size - 1 and decrement size.
- Remove: remove element at position $i$ and move all elements right of $i$ one position to the left, decrepement size.

# Amortized Analysis

- Worst-case analysis often pessimistic: append takes linear time if new memory allocated but in a sequence of append operations, this will happen rarely.
- Amortized analysis determines the average cost of an operation over an entire sequence of operations.
- Don't confuse this with an average-case analysis.
- Different methods
  - Aggregate analysis
  - Accounting method ← now
  - Potential method

# Accounting Method

- Assign charges to operations.
- Some operations charged more or less than they actually cost.
- If charged more: save difference as credit
- If charged less: use up some credit to pay for the difference.
- Credit must be non-negative all the time.
- Then the total amortized cost is always an upper bound on the actual total costs so far.

# Accounting Method: Append I

- Append without resize: constant cost (e.g. 1).
  Just insert the element at the right position.
- Append with resize: linear cost (1 for every element).
  - If the append element gets position $2^i$ ($i \in \mathbb{N}_{>0}$),
  - we first allocate overall space for $2^{i+1}$ elements, and
  - move all $2^i - 1$ existing elements to the new space.
- Starting from an empty array executing a sequence of append operations, we observe cost sequence
  1, 1, 3, 1, 5, 1, 1, 1, 9, 1, 1, 1, 1, 1, 1, 1, 17, 1 ...

# Accounting Method: Append II

Charge cost 3 for every append operation.

| size (after append) | capacity | charge | cost | credit |
|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 2 |
| 2 | 2 | 3 | 1 | 4 |
| 3 | 4 | 3 | 3 | 4 |
| 4 | 4 | 3 | 1 | 6 |
| 5 | 8 | 3 | 5 | 4 |
| 6 | 8 | 3 | 1 | 6 |
| 7 | 8 | 3 | 1 | 8 |
| 8 | 8 | 3 | 1 | 10 |
| 9 | 16 | 3 | 9 | 4 |
| 10 | 16 | 3 | 1 | 6 |

Charging 3 per operation covers all "running time costs".
$\rightarrow$ Append has constant amortized running time.

# Worst-Case Running Time Array

| Operation | Array |
|---|---|
| Access element by position | $O(1)$ |
| Prepend/remove first element | $O(n)$ |
| Append | $O(1)$ (amortized) |
| Remove last element | $O(1)$ (amortized) |
| Insert, remove from the middle | $O(n)$ |
| Traverse all elements | $O(n)$ |

# B1.3 Linked Lists

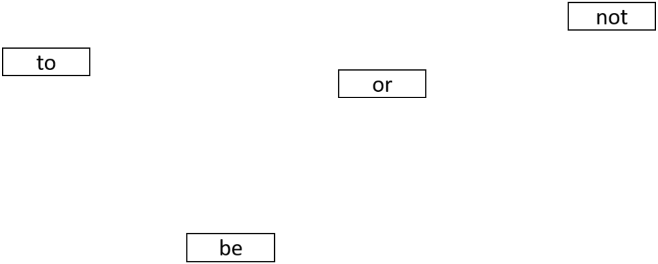
# Motivation

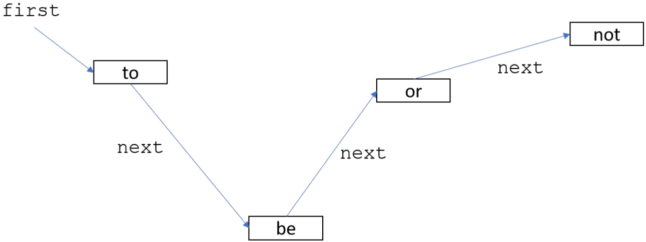
- Arrays need a large continuous block of memory.
- Inserting elements at arbitrary positions is expensive.

Alternative that allows us to distribute the elements in memory?

# Question?

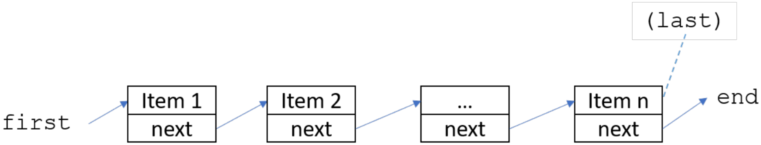
- How can we order elements that are distributed in memory?



# Linked Lists

- Every node stores its entry as well as a reference (pointer) to its successor.
- Need special value for the next pointer of the last element.
- ... or a reference to the last element.



# Jupyter Notebook

Jupyter notebook: `linked_lists.ipynb`

# Implementation: Node

```
class Node:
    def __init__(self, item, next=None):
        self.item = item
        self.next = next
```

# Implementation: List (without last reference)

```python
class LinkedList:
    def __init__(self):
        self.first = None

    # prepend item at the front of the list
    def prepend(self, item):
        new_node = Node(item, self.first)
        self.first = new_node

    ... # other methods added to notebook after lecture
```

# Worst-Case Running Time Array / Linked List

| Operation | Array | Linked List |
|---|---|---|
| Prepend/remove first element | $O(n)$ | $O(1)$ |
| Append | $O(1)$ (amortized) | $O(n)$ |
| Remove last element | $O(1)$ (amortized) | $O(n)$ |
| Insert, remove from the middle | $O(n)$ | $O(n)$ |
| Traverse all elements | $O(n)$ | $O(n)$ |
| Find an element | $O(n)$ | $O(n)$ |
| Access element by position | $O(1)$ | – |

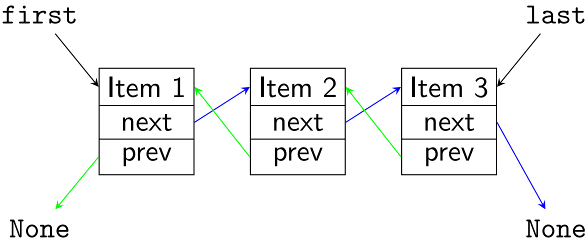
What running times could we improve if we also maintained a pointer to the last element of the linked list?

**Take-home Message**

- Different data structures have different trade-offs.

# Doubly Linked Lists

- Idea: Do not only store a reference to the successor but also to the predecessor.
- Renders appending at/removal from end constant time.



# Jupyter Notebook

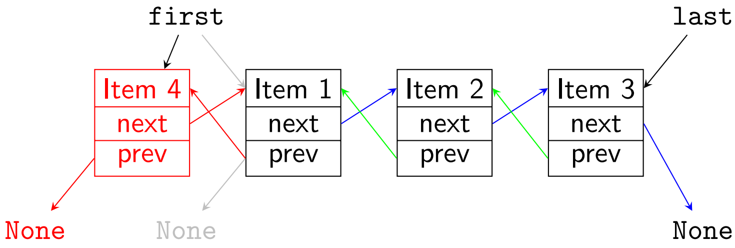
Jupyter notebook: `doubly_linked_lists.ipynb`

# Doubly Linked Lists: Implementation

```python
class Node:
    def __init__(self, item, next=None):
        self.item = item
        self.next = next

class DoublyLinkedList:
    def __init__(self):
        self.first = None
        self.last = None

    def is_empty(self):
        return self.first is None

    # other methods on next slides
```

# Doubly Linked Lists: prepend

```
    def prepend(self, item):
        if self.is_empty():
            self.first = Node(item)
            self.last = self.first
        else:
            node = Node(item, self.first, None)
            self.first.prev = node
            self.first = node
```



# Doubly Linked Lists: append

```
    def append(self, item):
        if self.is_empty():
            self.first = Node(item)
            self.last = self.first
        else:
            node = Node(item, None, self.last)
            self.last.next = node
            self.last = node
```

# Doubly Linked Lists: remove_first

```
    def remove_first(self):
        if self.is_empty():
            raise Exception("removing from empty list")
        item = self.first.item
        self.first = self.first.next
        if self.first is not None:
            self.first.prev = None
        else:
            self.last = None
        return item
```

# Doubly Linked Lists: remove_last

With doubly linked lists, removing the last element is analogous to removing the first element:

```
    def remove_last(self):
        if self.is_empty():
            raise Exception("removing from empty list")
        item = self.last.item
        self.last = self.last.prev
        if self.last is not None:
            self.last.next = None
        else:
            self.first = None
        return item
```

# Worst-Case Running Time Array / Doubly Linked List

| Operation | Array | Doubly Linked List |
|---|---|---|
| Prepend/remove first element | $O(n)$ | $O(1)$ |
| Append | $O(1)$ (amort.) | $O(1)$ |
| Remove last element | $O(1)$ (amort.) | $O(1)$ |
| Insert, remove in the middle | $O(n)$ | $O(n)/O(1)^*$ |
| Traverse all elements | $O(n)$ | $O(n)$ |
| Find an element | $O(n)$ | $O(n)$ |
| Access element by position | $O(1)$ | – |

* constant, if node at the position is parameter

### Take-home Message

Compared to singly linked lists, doubly linked lists need a linear amount of additional memory (for the `prev` references), but provide better running times for operations at the end of the list.

# B1.4 Summary

# Summary

- An amortized analysis determines the average cost of an operation over an entire sequence of operations.
- Arrays and linked lists store sequences of items.
  - Arrays store items in a continuous space and can efficiently access an item by index.
  - Linked lists store items in nodes with a reference to the next node (doubly linked lists: also to previous node).