Algorithms and Data Structures

A6. Runtime Analysis: Logarithm

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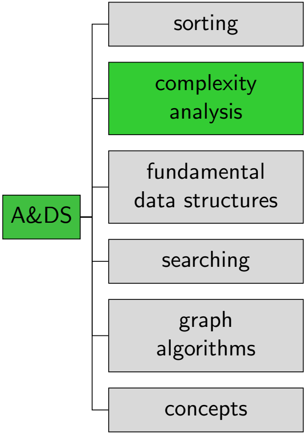
Logarithm

Content of the Course



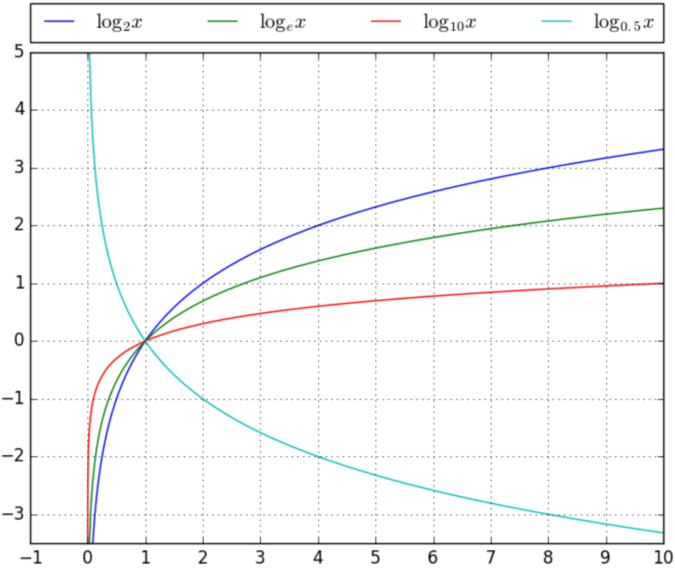
Logarithm

- For the analysis of merge sort, we will need the logarithm function.
- This is often the case in runtime analysis, in particular for divide-and-conquer algorithms.
- The logarithm to the base b is the inverse function to exponentiation with base b, i.e.

$$\log_b x = y \text{ iff. } b^y = x.$$

- Example: $\log_2 8 = 3$, because $2^3 = 8$
  Example: $\log_3 81 = 4$, because $3^4 = 81$
- $\log_b a$ intuitively (if this works without remainder):
  "How often must we divide a by b to reach 1?"

Logarithm: Illustration



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product	$\log_b(xy) = \log_b x + \log_b y$
power	$\log_b(x^r) = r \log_b x$
change of base	$\log_b x = \log_a x / \log_a b$

Logarithm: Example Calculation

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$$\begin{aligned}
5^{\log_2 x} &= \left(2^{\log_2 5}\right)^{\log_2 x} \\
&= 2^{\log_2 5 \log_2 x} \\
&= 2^{\log_2 x \log_2 5} \\
&= \left(2^{\log_2 x}\right)^{\log_2 5} \\
&= x^{\log_2 5} \\
&\approx x^{2.32}
\end{aligned}$$