

Algorithms and Data Structures

A4. Sorting II: Merge Sort

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February 29/March 6, 2024

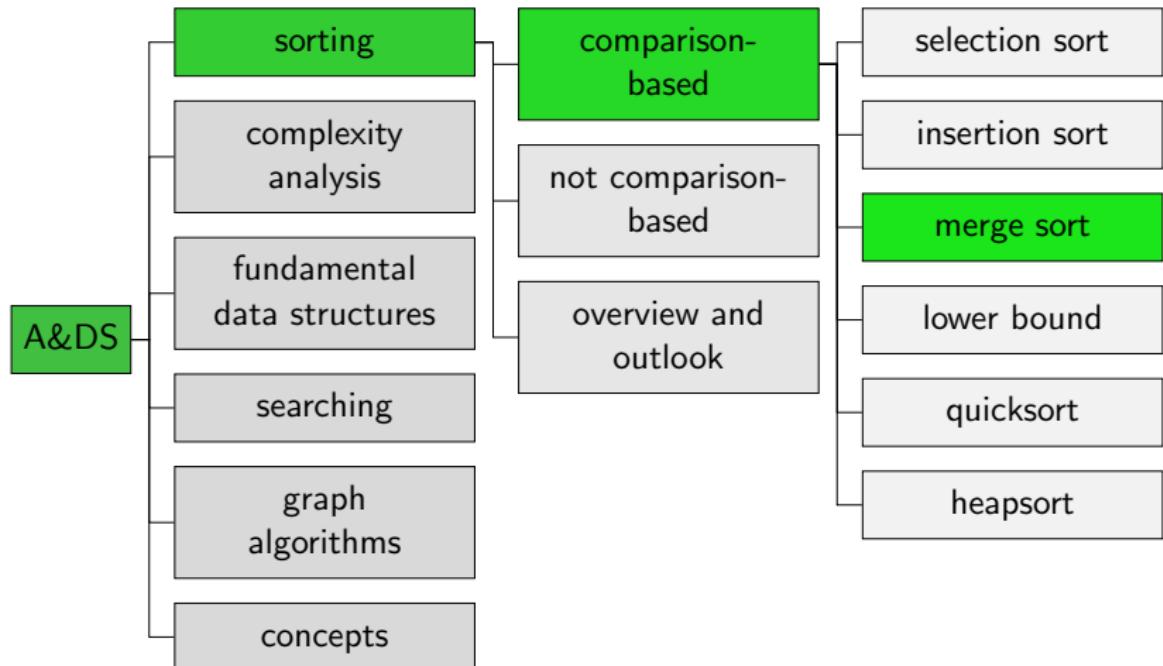
Merge Sort

Merge Step

Top-Down Merge Sort

Merge Sort

Content of the Course



Merge Sort: Idea

- **Observation:** two sorted sequences can easily be combined to a single sorted sequence.
- Empty sequences or sequences with a single element are sorted.
- **Idea** for longer sequences:
 - divide the input sequence into two roughly equally-sized ranges
 - recursive call for each of the two ranges
 - merge now sorted ranges into one
- divide-and-conquer approach

Merge Sort
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Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Merge Sort: Illustration

7 3 2 9 7 1 4 5

Merge Sort
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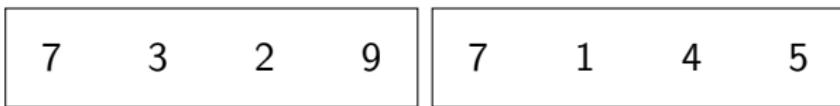
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Merge Sort: Illustration



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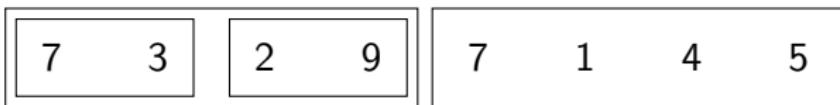
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Bottom-Up Merge Sort
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Merge Sort: Illustration



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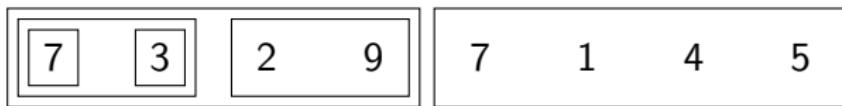
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Merge Sort: Illustration



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Bottom-Up Merge Sort
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Merge Sort: Illustration



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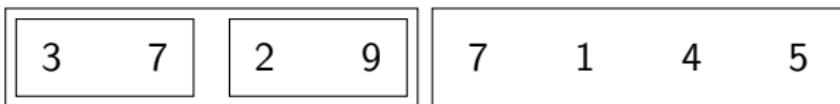
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Bottom-Up Merge Sort
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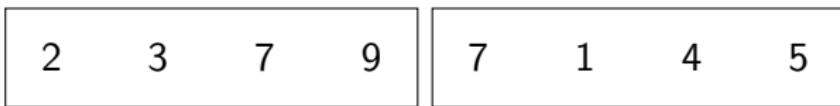
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Merge Sort: Illustration



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Bottom-Up Merge Sort
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Merge Sort: Illustration



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Merge Sort: Illustration



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Merge Sort: Illustration



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Merge Sort: Illustration



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Merge Sort: Illustration



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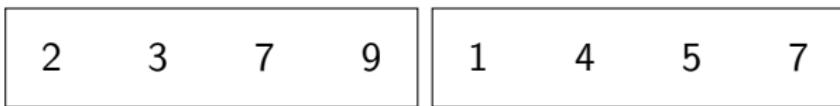
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Merge Sort: Illustration



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Bottom-Up Merge Sort
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Summary
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Merge Sort: Illustration

1 2 3 4 5 7 7 9

Merge Sort
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Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Merge Step

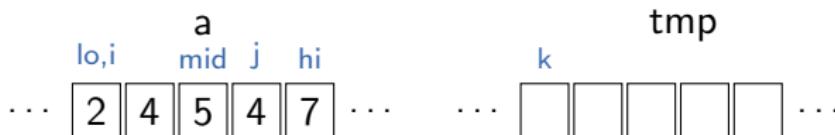
Merging the Sorted Ranges

- indices $lo \leq mid < hi$
- **prerequisite:** array[lo] to array[mid] and array[$mid+1$] to array[hi] already sorted
- **aim:** array[lo] to array[hi] sorted
- **idea:** process both ranges in parallel from front to end and collect the smaller element
- use additional storage for the collected entries

Merge Step: Example

Array tmp has same size as input array.

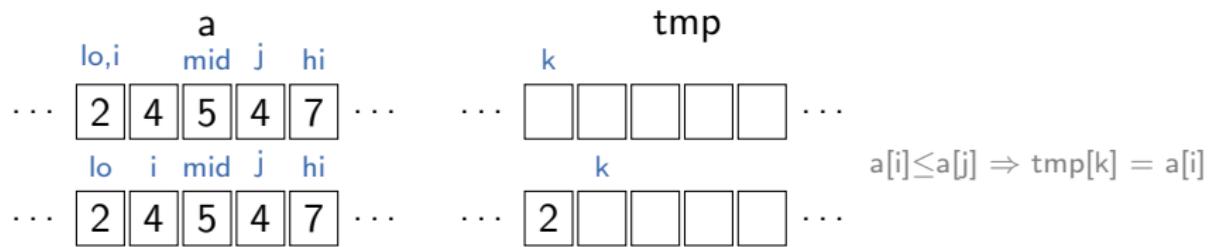
initialize: $i := lo$, $j := mid + 1$, $k := lo$



Merge Step: Example

Array tmp has same size as input array.

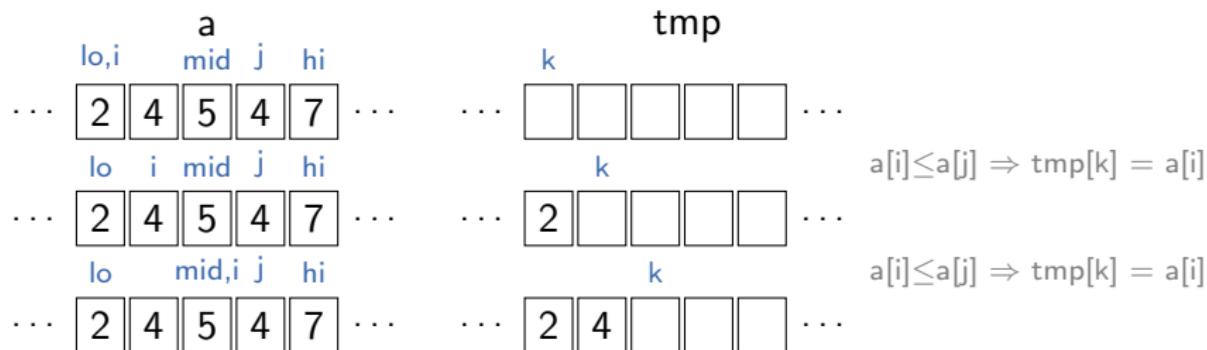
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Merge Step: Example

Array tmp has same size as input array.

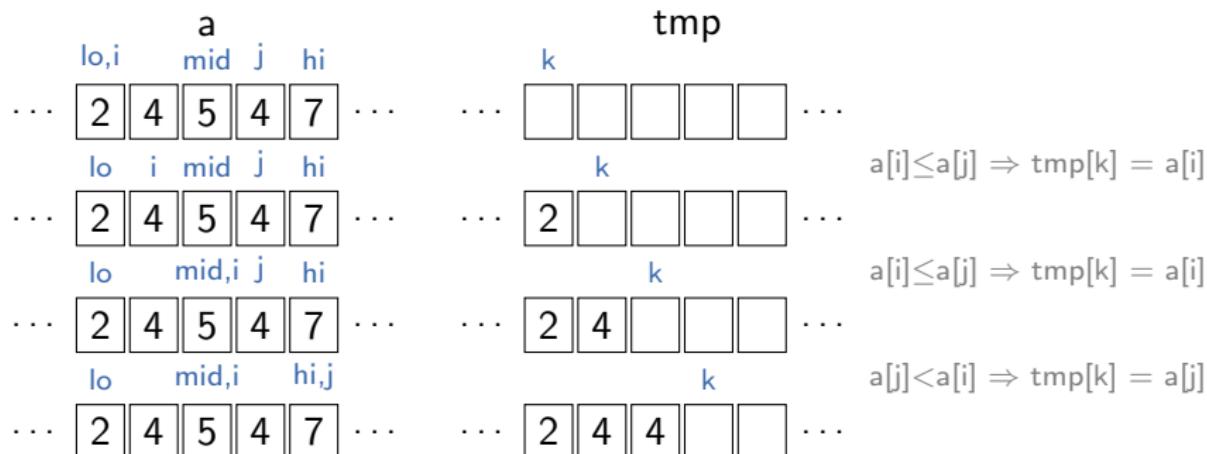
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Merge Step: Example

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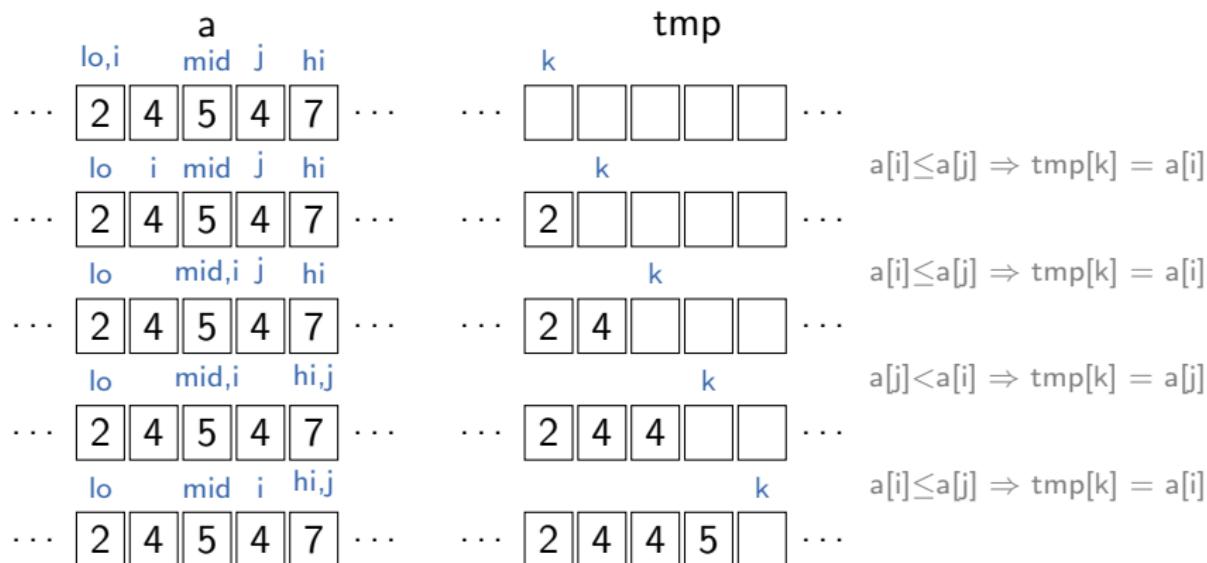
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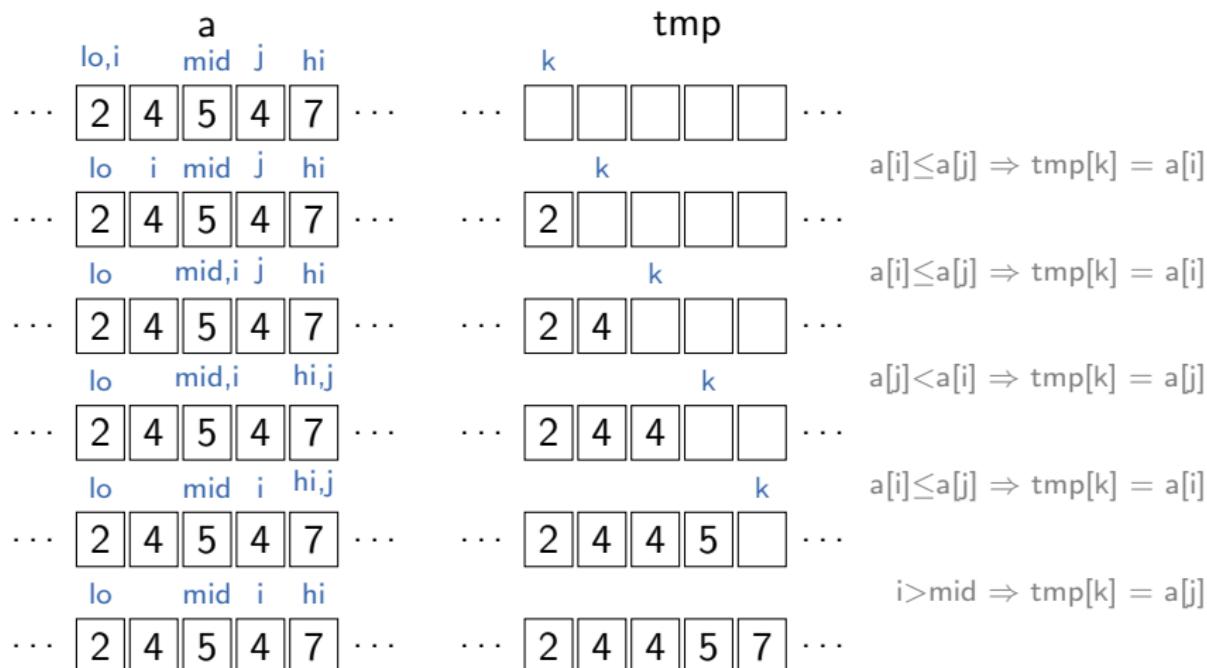
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Merge Step: Example

Array tmp has same size as input array.

initialize: $i := lo$, $j := mid + 1$, $k := lo$



Merge Step: Algorithm

```
1 def merge(array, tmp, lo, mid, hi):
2     i = lo
3     j = mid + 1
4     for k in range(lo, hi + 1):  # k = lo, ..., hi
5         if j > hi or (i <= mid and array[i] <= array[j]):
6             tmp[k] = array[i]
7             i += 1
8         else:
9             tmp[k] = array[j]
10            j += 1
11     for k in range(lo, hi + 1):  # k = lo, ..., hi
12         array[k] = tmp[k]
```

Merge Step: Algorithm

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11     for k in range(lo, hi + 1):  # k = lo, ..., hi
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```

Also correct for $lo = mid = hi$

Merge Sort
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Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Jupyter Notebook



Jupyter notebook: `merge_sort.ipynb`

Merge Sort
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Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Questions



Merge Sort
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Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Top-Down Merge Sort

Merge Sort: Algorithm

recursive top-down variant

```
1 def sort(array):  
2     tmp = [0] * len(array)  # [0, ..., 0] with same size as array  
3     sort_aux(array, tmp, 0, len(array) - 1)  
4  
5 def sort_aux(array, tmp, lo, hi):  
6     if hi <= lo:  
7         return  
8     mid = lo + (hi - lo) // 2  
9     # //: floor division  
10    sort_aux(array, tmp, lo, mid)  
11    sort_aux(array, tmp, mid + 1, hi)  
12    merge(array, tmp, lo, mid, hi)
```

Merge Sort
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Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Possible Improvements

- on short sequences, insertion sort faster than merge sort
→ use insertion sort for small `hi - lo`

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    return
```

Possible Improvements

- on short sequences, insertion sort faster than merge sort
→ use insertion sort for small `hi - lo`
- directly skip the merge step if positions `lo` to `hi` already sorted

```
if array[mid] <= array[mid + 1]:  
    return
```
- copying `tmp` in merge takes time
→ swap role of `array` and `tmp` in every recursive call

Merge Step: Correctness

- **Invariant:** at the end of each iteration of the loop:
 - $\text{tmp}[k] \leq \text{array}[m]$ for all $i \leq m \leq \text{mid}$, and
 - $\text{tmp}[k] \leq \text{array}[n]$ for all $j \leq n \leq \text{hi}$.
- tmp is written from left to right.
- After the last iteration of the loop it holds for all $\text{lo} \leq r < s \leq \text{hi}$ that $\text{tmp}[r] \leq \text{tmp}[s]$ ($=$ range is sorted).

Merge Sort: Correctness

sort_aux:

- Proof by induction over length $hi - lo$
(always 1 smaller than the number of cells in the range)
- Basis $hi - lo = -1$: empty range is sorted.
- Basis $hi - lo = 0$: range with a single element is sorted.

Merge Sort: Correctness

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- Proof by induction over length $hi - lo$
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- Induction hypothesis: merge sort is correct for all $hi - lo < m$
- Inductive step ($m - 1 \rightarrow m$):

Merge Sort: Correctness

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- Inductive step ($m - 1 \rightarrow m$):
Merge sort makes two recursive calls with $hi - lo \leq \lfloor m/2 \rfloor$,
afterwards the input is sorted **between lo and mid** and
between mid + 1 and hi. (by ind. hyp.)

Merge Sort: Correctness

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Since the merge step is correct, at the end the entire range **from lo to hi is sorted**.

Merge Sort: Correctness

`sort_aux:`

- Proof by induction over length $hi - lo$
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afterwards the input is sorted **between lo and mid** and
between $mid + 1$ and hi . (by ind. hyp.)

Since the merge step is correct, at the end the entire range
from lo to hi is sorted.

Merge sort: calls `sort_aux` for the entire range of the input,
thus at the end the entire input has been sorted.

Merge Sort: Properties (Slido)

```
1 def sort(array):  
2     tmp = [0] * len(array)  # [0,...,0] with same size as array  
3     sort_aux(array, tmp, 0, len(array) - 1)  
4  
5 def sort_aux(array, tmp, lo, hi):  
6     if hi <= lo:  
7         return  
8     mid = lo + (hi - lo) // 2  
9     # //: floor division  
10    sort_aux(array, tmp, lo, mid)  
11    sort_aux(array, tmp, mid + 1, hi)  
12    merge(array, tmp, lo, mid, hi)
```

Which of the following properties does merge sort have? In-place? Adaptive? Stable?



Merge Sort: Properties

- **not in-place**: uses non-constant storage for `tmp` and call stack
- **running time**: not adaptive
(except with merge-skipping improvement)
precise analysis: [later chapter](#)
- **stable**: merge prefers `array[i]` if `array[i]` equals `array[j]`.

Merge Sort
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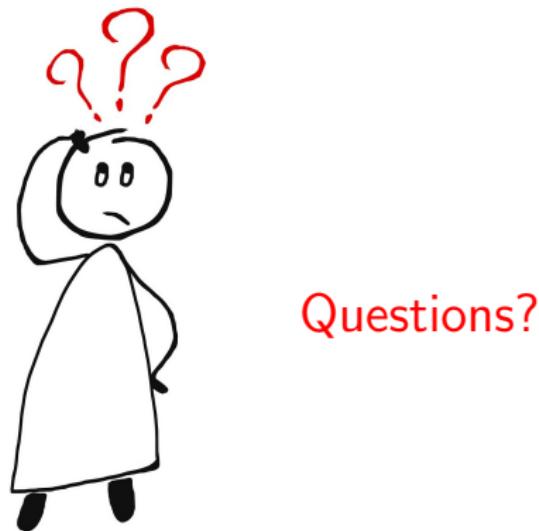
Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Questions



Merge Sort
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Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Bottom-Up Merge Sort

Merge Sort
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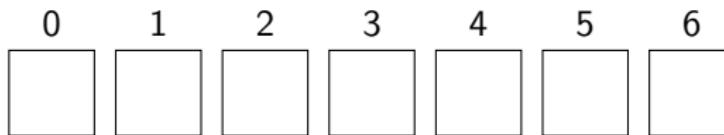
Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Bottom-Up Variant



Merge Sort
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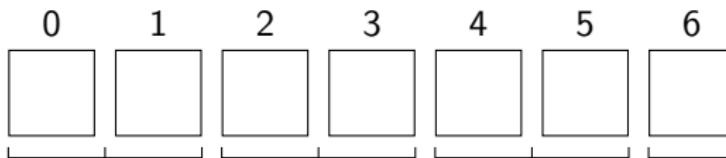
Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Bottom-Up Variant



Merge Sort
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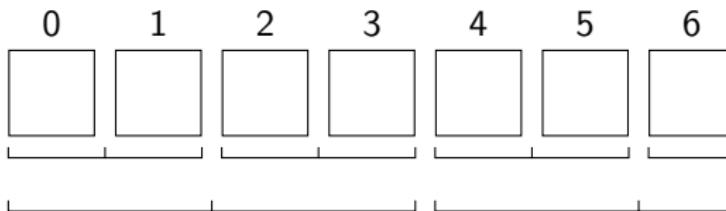
Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Bottom-Up Variant



Merge Sort
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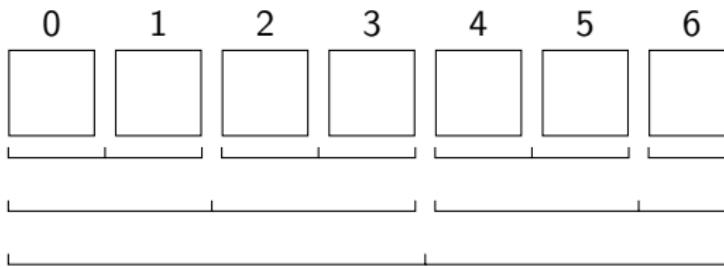
Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Bottom-Up Variant



Merge Sort
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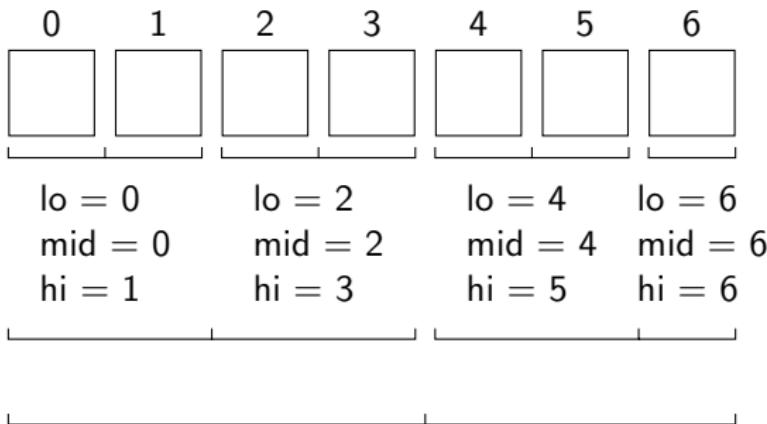
Merge Step
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Top-Down Merge Sort
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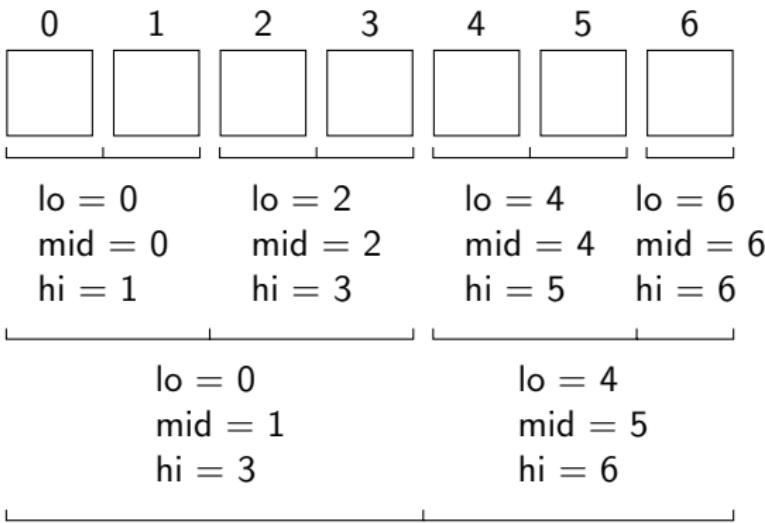
Bottom-Up Merge Sort
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Summary
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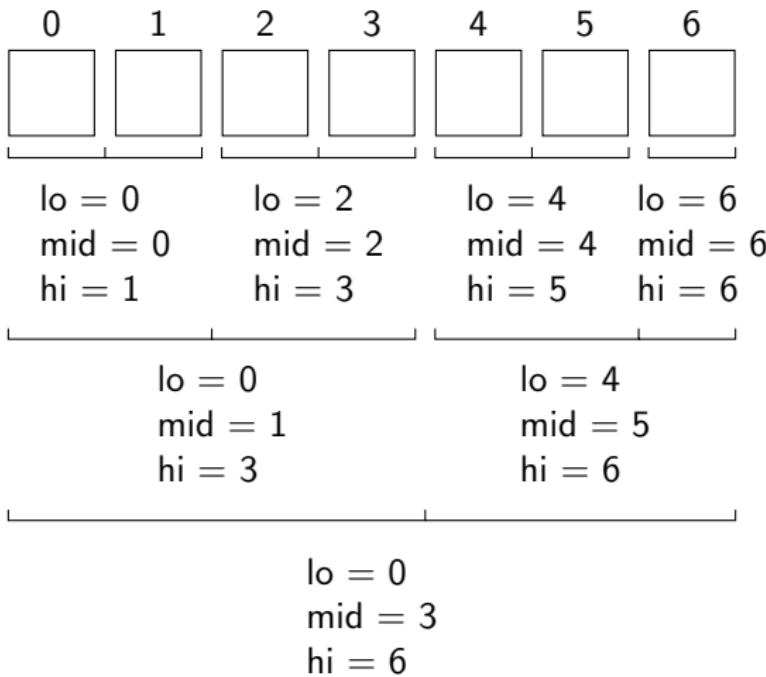
Bottom-Up Variant



Bottom-Up Variant



Bottom-Up Variant



Bottom-Up Merge Sort: Algorithm

iterative bottom-up variant

```
1 def sort(array):  
2     n = len(array)  
3     tmp = [0] * n  
4     length = 1  
5     while length < n:  
6         lo = 0  
7         while lo < n - length:  
8             mid = lo + length - 1  
9             hi = min(lo + 2 * length - 1, n - 1)  
10            merge(array, tmp, lo, mid, hi)  
11            lo += 2 * length  
12            length *= 2
```

Merge Sort
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Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary
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Questions



Merge Sort
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Merge Step
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Top-Down Merge Sort
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Bottom-Up Merge Sort
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Summary

Summary

- Merge sort is a **divide-and-conquer** algorithm, which divides the input sequence into two roughly equally-sized ranges.
- The **merge step** combines two already sorted ranges.
- Merge sort is **stable**, but does **not work in-place**.
- The **top-down variant** is a **recursive** algorithm.
- The **bottom-up variant** is an **iterative** algorithm.