

# Algorithms and Data Structures

## A4. Sorting II: Merge Sort

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February 29/March 6, 2024

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A4.1 Merge Sort

A4.2 Merge Step

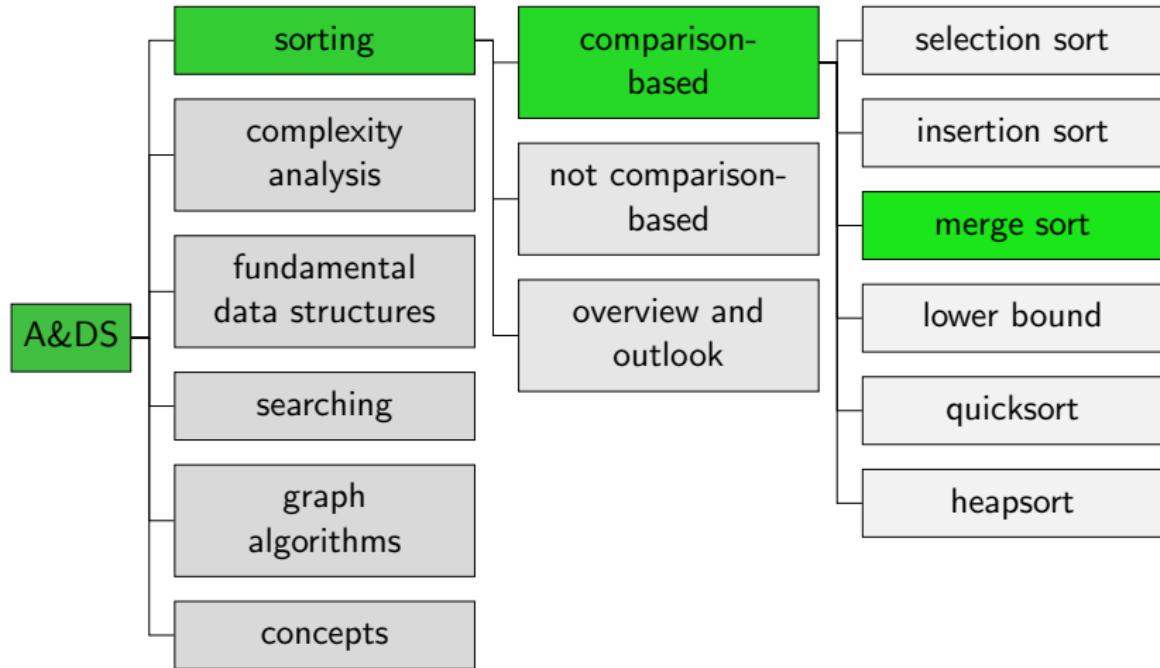
A4.3 Top-Down Merge Sort

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## A4.1 Merge Sort

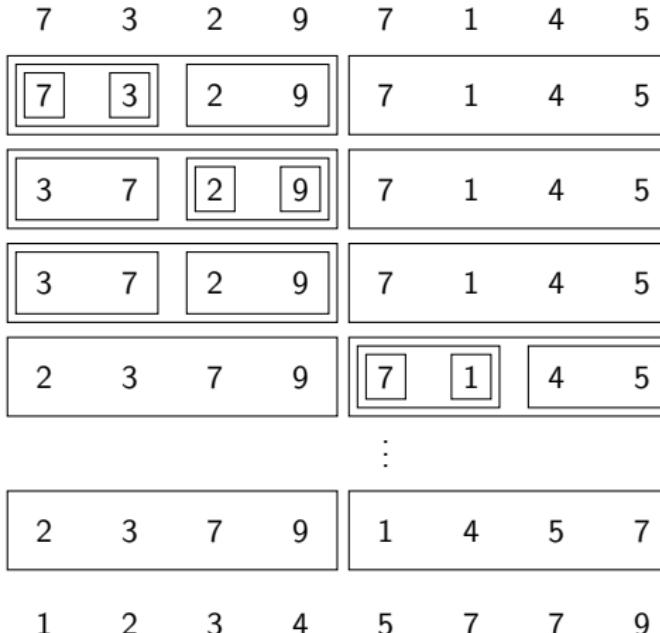
# Content of the Course



## Merge Sort: Idea

- ▶ **Observation:** two sorted sequences can easily be combined to a single sorted sequence.
- ▶ Empty sequences or sequences with a single element are sorted.
- ▶ **Idea** for longer sequences:
  - ▶ divide the input sequence into two roughly equally-sized ranges
  - ▶ recursive call for each of the two ranges
  - ▶ merge now sorted ranges into one
- ▶ **divide-and-conquer approach**

## Merge Sort: Illustration



(Detailed animation in screen version of slides)

## A4.2 Merge Step

## Merging the Sorted Ranges

- ▶ indices  $lo \leq mid < hi$
- ▶ **prerequisite:** array[ $lo$ ] to array[ $mid$ ] and array[ $mid+1$ ] to array[ $hi$ ] already sorted
- ▶ **aim:** array[ $lo$ ] to array[ $hi$ ] sorted
- ▶ **idea:** process both ranges in parallel from front to end and collect the smaller element
- ▶ use additional storage for the collected entries

# Merge Step: Example

Array tmp has same size as input array.

initialize:  $i := \text{lo}$ ,  $j := \text{mid} + 1$ ,  $k := \text{lo}$

$a$	$tmp$	
$\text{lo}, i$ $2 \boxed{4} \boxed{5} \boxed{4} \boxed{7} \dots$	$k$ $\boxed{\phantom{0}} \boxed{\phantom{0}} \boxed{\phantom{0}} \boxed{\phantom{0}} \boxed{\phantom{0}} \dots$	
$\text{lo} \quad i \quad \text{mid} \quad j \quad \text{hi}$	$k$	$a[i] \leq a[j] \Rightarrow \text{tmp}[k] = a[i]$
$\dots \quad 2 \boxed{4} \boxed{5} \boxed{4} \boxed{7} \dots$	$\boxed{2} \boxed{\phantom{0}} \boxed{\phantom{0}} \boxed{\phantom{0}} \boxed{\phantom{0}} \dots$	
$\text{lo} \quad \text{mid}, i \quad j \quad \text{hi}$	$k$	$a[i] \leq a[j] \Rightarrow \text{tmp}[k] = a[i]$
$\dots \quad 2 \boxed{4} \boxed{5} \boxed{4} \boxed{7} \dots$	$\boxed{2} \boxed{4} \boxed{\phantom{0}} \boxed{\phantom{0}} \boxed{\phantom{0}} \dots$	
$\text{lo} \quad \text{mid}, i \quad \text{hi}, j$	$k$	$a[j] < a[i] \Rightarrow \text{tmp}[k] = a[j]$
$\dots \quad 2 \boxed{4} \boxed{5} \boxed{4} \boxed{7} \dots$	$\boxed{2} \boxed{4} \boxed{4} \boxed{\phantom{0}} \boxed{\phantom{0}} \dots$	
$\text{lo} \quad \text{mid} \quad i \quad \text{hi}, j$	$k$	$a[i] \leq a[j] \Rightarrow \text{tmp}[k] = a[i]$
$\dots \quad 2 \boxed{4} \boxed{5} \boxed{4} \boxed{7} \dots$	$\boxed{2} \boxed{4} \boxed{4} \boxed{5} \boxed{\phantom{0}} \dots$	
$\text{lo} \quad \text{mid} \quad i \quad \text{hi}$		$i > \text{mid} \Rightarrow \text{tmp}[k] = a[j]$
$\dots \quad 2 \boxed{4} \boxed{5} \boxed{4} \boxed{7} \dots$	$\boxed{2} \boxed{4} \boxed{4} \boxed{5} \boxed{7} \dots$	

## Merge Step: Algorithm

---

```
1 def merge(array, tmp, lo, mid, hi):
2     i = lo
3     j = mid + 1
4     for k in range(lo, hi + 1):  # k = lo, ..., hi
5         if j > hi or (i <= mid and array[i] <= array[j]):
6             tmp[k] = array[i]
7             i += 1
8         else:
9             tmp[k] = array[j]
10            j += 1
11     for k in range(lo, hi + 1):  # k = lo, ..., hi
12         array[k] = tmp[k]
```

---

Also correct for  $lo = mid = hi$

# Jupyter Notebook



Jupyter notebook: `merge_sort.ipynb`

## A4.3 Top-Down Merge Sort

# Merge Sort: Algorithm

recursive top-down variant

---

```
1 def sort(array):  
2     tmp = [0] * len(array)  # [0,...,0] with same size as array  
3     sort_aux(array, tmp, 0, len(array) - 1)  
4  
5 def sort_aux(array, tmp, lo, hi):  
6     if hi <= lo:  
7         return  
8     mid = lo + (hi - lo) // 2  
9     # //: floor division  
10    sort_aux(array, tmp, lo, mid)  
11    sort_aux(array, tmp, mid + 1, hi)  
12    merge(array, tmp, lo, mid, hi)
```

---

## Possible Improvements

- ▶ on short sequences, insertion sort faster than merge sort  
→ use insertion sort for small `hi - lo`
- ▶ directly skip the merge step if positions `lo` to `hi` already sorted

```
if array[mid] <= array[mid + 1]:  
    return
```

- ▶ copying `tmp` in merge takes time  
→ swap role of `array` and `tmp` in every recursive call

## Merge Step: Correctness

- ▶ **Invariant:** at the end of each iteration of the loop:
  - ▶  $\text{tmp}[k] \leq \text{array}[m]$  for all  $i \leq m \leq \text{mid}$ , and
  - ▶  $\text{tmp}[k] \leq \text{array}[n]$  for all  $j \leq n \leq \text{hi}$ .
- ▶  $\text{tmp}$  is written from left to right.
- ▶ After the last iteration of the loop it holds for all  $\text{lo} \leq r < s \leq \text{hi}$  that  $\text{tmp}[r] \leq \text{tmp}[s]$  ( $=$  range is sorted).

# Merge Sort: Correctness

`sort_aux`:

- ▶ Proof by induction over length  $hi - lo$   
(always 1 smaller than the number of cells in the range)
- ▶ Basis  $hi - lo = -1$ : empty range is sorted.
- ▶ Basis  $hi - lo = 0$ : range with a single element is sorted.
- ▶ Induction hypothesis: merge sort is correct for all  $hi - lo < m$
- ▶ Inductive step ( $m - 1 \rightarrow m$ ):

Merge sort makes two recursive calls with  $hi - lo \leq \lfloor m/2 \rfloor$ ,  
afterwards the input is sorted **between  $lo$  and  $mid$**  and  
**between  $mid + 1$  and  $hi$** . (by ind. hyp.)

Since the merge step is correct, at the end the entire range  
from  $lo$  to  $hi$  is sorted.

**Merge sort:** calls `sort_aux` for the entire range of the input,  
thus at the end the entire input has been sorted.

# Merge Sort: Properties (Slido)

---

```
1 def sort(array):  
2     tmp = [0] * len(array)  # [0, ..., 0] with same size as array  
3     sort_aux(array, tmp, 0, len(array) - 1)  
4  
5 def sort_aux(array, tmp, lo, hi):  
6     if hi <= lo:  
7         return  
8     mid = lo + (hi - lo) // 2  
9     # //: floor division  
10    sort_aux(array, tmp, lo, mid)  
11    sort_aux(array, tmp, mid + 1, hi)  
12    merge(array, tmp, lo, mid, hi)
```

---

Which of the following properties does merge sort have? In-place? Adaptive? Stable?

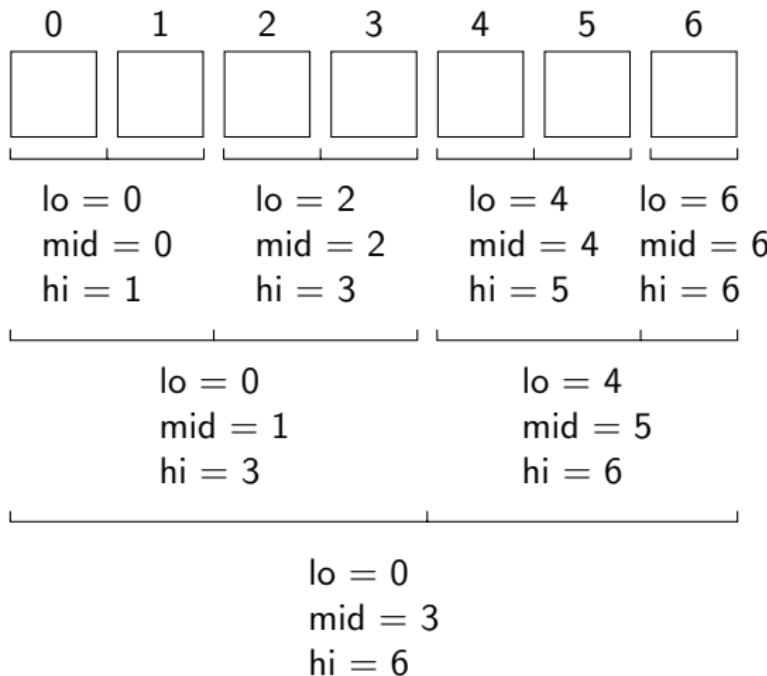


# Merge Sort: Properties

- ▶ **not in-place**: uses non-constant storage for `tmp` and call stack
- ▶ **running time**: not adaptive  
(except with merge-skipping improvement)  
*precise analysis: later chapter*
- ▶ **stable**: merge prefers `array[i]` if `array[i]` equals `array[j]`.

## A4.4 Bottom-Up Merge Sort

## Bottom-Up Variant



# Bottom-Up Merge Sort: Algorithm

iterative bottom-up variant

---

```
1 def sort(array):  
2     n = len(array)  
3     tmp = [0] * n  
4     length = 1  
5     while length < n:  
6         lo = 0  
7         while lo < n - length:  
8             mid = lo + length - 1  
9             hi = min(lo + 2 * length - 1, n - 1)  
10            merge(array, tmp, lo, mid, hi)  
11            lo += 2 * length  
12            length *= 2
```

---

# A4.5 Summary

# Summary

- ▶ Merge sort is a **divide-and-conquer** algorithm, which divides the input sequence into two roughly equally-sized ranges.
- ▶ The **merge step** combines two already sorted ranges.
- ▶ Merge sort is **stable**, but does **not work in-place**.
- ▶ The **top-down variant** is a **recursive** algorithm.
- ▶ The **bottom-up variant** is an **iterative** algorithm.