# Algorithms and Data Structures

## A3. Sorting I: Selection and Insertion Sort

Gabriele Röger

University of Basel

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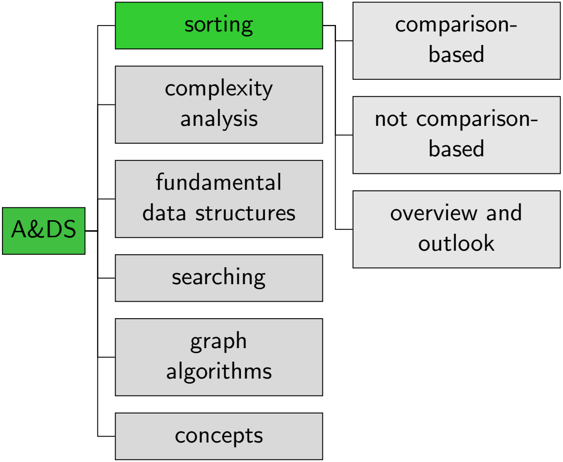
# Sorting

# Content of the Course



# Relevance

sorting data important for many applications, such as

- sorted presentation (e.g. on website)
  - products sorted by price, rating, ...
  - account transactions sorted by transaction date
- preprocessing for many efficient search algorithms
  - How quickly can you find a number in a (physical) telephone book? How quickly could you do so if the entries were not sorted?
- subroutine of many other algorithms
  - e.g. a program that renders layered graphical objects might sort them to determine where objects are covered by other objects

Journal "Computing in Science & Engineering" lists Quicksort as one of the 10 most important algorithms of the 20th century.

# Sorting Problem

## Sorting Problem

Input

- sequence of $n$ elements $e_1, \dots, e_n$
- each element $e_i$ has key $k_i = key(e_i)$
- partial order $\leq$ on the keys
  reflexive: $k \leq k$
  transitive: $k \leq k'$ and $k' \leq k'' \Rightarrow k \leq k''$
  antisymmetric: $k \leq k'$ and $k' \leq k \Rightarrow k = k'$

Output

- Sequence of the same elements sorted according to the ordering relation on its keys

Notation: also $e \leq e'$ for $key(e) \leq key(e')$

# Sorting Problem: Examples

**Example**

**Input:** $\langle 3, 6, 2, 3, 1\rangle$, $key(e) = e$, $\leq$ on the integers
**Output:** $\langle 1, 2, 3, 3, 6\rangle$

**Example**

**Input:** list of all students of the Univ. of Basel,
$key(e) = \langle$place of residence of $e\rangle$, lexicographic order
**Output:** list of all students, sorted by their place of residence

Is the output uniquely defined?

In this course: mostly integers, $key(e) = e$ and $\leq$ on integers

# Interesting Properties of Sorting Algorithms

- running time: how many key comparisons and swaps of elements are executed? adaptive: algorithms faster if input already (partially) sorted

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  in-place: needs no additional storage beyond the input array and a constant amount of space
  (independent of the input size)

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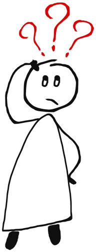
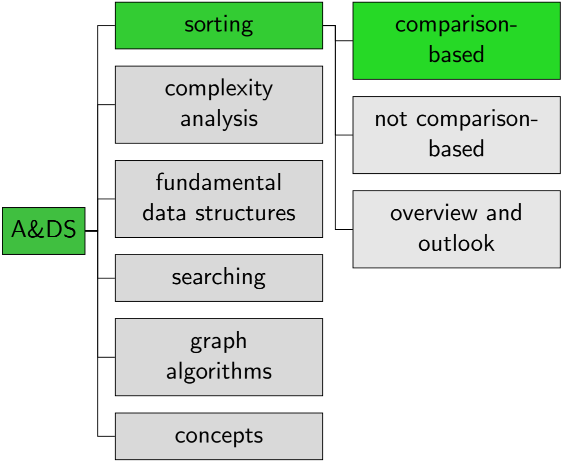
- stable: elements with the same value appear in the output sequence in the same order as they do in the input sequence
- comparison-based: uses only key comparisons and swaps of elements

# Questions



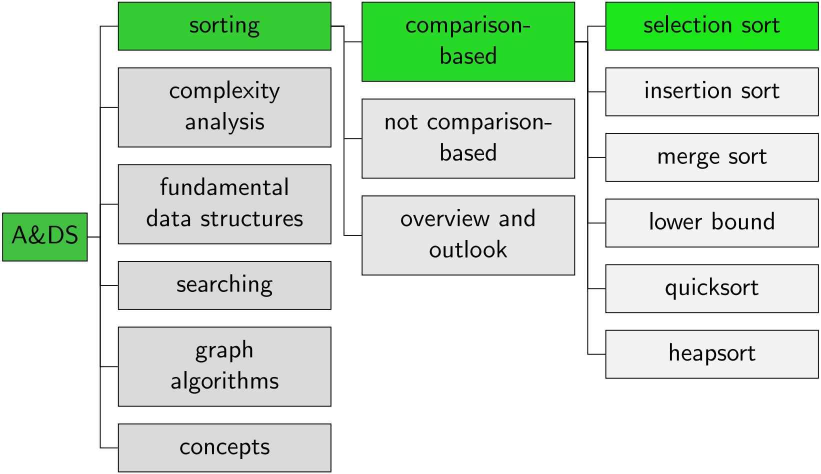
Questions?

# Content of the Course

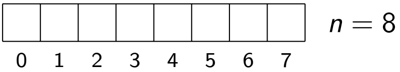


# Selection Sort

# Content of the Course



# Selection Sort: Informally




- identify smallest element at positions $0, \dots, n-1$ and swap it to position 0
- identify smallest element at positions $1, \dots, n-1$ and swap it to position 1
- ...
- identify smallest element at positions $n-2, n-1$ and swap it to position n - 2

# Selection Sort: Example

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |

# Selection Sort: Example

| | | | | | | | |
|---|---|---|---|---|---|---|---|
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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 7 | 2 | 9 | 7 | 3 | 4 | 5 |

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 7 | 9 | 7 | 3 | 4 | 5 |

# Selection Sort: Example

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |
| 1 | 7 | 2 | 9 | 7 | 3 | 4 | 5 |
| 1 | 2 | 7 | 9 | 7 | 3 | 4 | 5 |
| 1 | 2 | 3 | 9 | 7 | 7 | 4 | 5 |
| 1 | 2 | 3 | 4 | 7 | 7 | 9 | 5 |
| 1 | 2 | 3 | 4 | 5 | 7 | 9 | 7 |
| 1 | 2 | 3 | 4 | 5 | 7 | 9 | 7 |
| 1 | 2 | 3 | 4 | 5 | 7 | 7 | 9 |

# Selection Sort: Algorithm

```
def selection_sort(array):
    n = len(array)
    for i in range(n - 1):  # i = 0, ..., n-2
        # find index of minimum element at positions i, ..., n-1
        min_index = i
        for j in range(i + 1, n):  # j = i+1, ..., n-1
            if array[j] < array[min_index]:
                min_index = j
        # swap element at position i with minimum element
        array[i], array[min_index] = array[min_index], array[i]
```

# Selection Sort: Example

| i | min_ind. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| | | 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |
| 0 | 5 | 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |

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| 0 | 5 | 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |
| 1 | 2 | 1 | 7 | 2 | 9 | 7 | 3 | 4 | 5 |
| 2 | 5 | 1 | 2 | 7 | 9 | 7 | 3 | 4 | 5 |

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| 0 | 5 | 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |
| 1 | 2 | 1 | 7 | 2 | 9 | 7 | 3 | 4 | 5 |
| 2 | 5 | 1 | 2 | 7 | 9 | 7 | 3 | 4 | 5 |
| 3 | 6 | 1 | 2 | 3 | 9 | 7 | 7 | 4 | 5 |
| 4 | 7 | 1 | 2 | 3 | 4 | 7 | 7 | 9 | 5 |
| 5 | 5 | 1 | 2 | 3 | 4 | 5 | 7 | 9 | 7 |
| 6 | 7 | 1 | 2 | 3 | 4 | 5 | 7 | 9 | 7 |
| | | 1 | 2 | 3 | 4 | 5 | 7 | 7 | 9 |

looking for minimum among dark entries

red entry is found minimum

gray entries already sorted

# Correctness

## Correctness of an algorithm

An algorithm for a computational problem is correct if for every problem instance provided as input, it

- halts, i.e. it finishes its computation in finite time, and
- determines a correct solution to the problem instance.

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- correctness of invariants by (joint) induction
- after the last iteration, all elements except for the last one are in the correct order and the last one is not smaller than the second-last. $\rightarrow$ entire sequence sorted

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- Invariant 2: at the end of each iteration of the outer loop, none of the elements at positions $\leq i$ is (strictly) larger than an element at a position $> i$.
- correctness of invariants by (joint) induction
- after the last iteration, all elements except for the last one are in the correct order and the last one is not smaller than the second-last. $\rightarrow$ entire sequence sorted
- Termination: $n-1$ iterations of outer loop, each with fewer than $n$ iterations of inner loop $\rightarrow$ finite runtime

# Properties of Selection Sort

- in-place: additional storage does not depend on input size
- running time: does only depend on the size of the input (not adaptive)
  exact analysis: later chapter
- not stable: can swap the element at position $i$ behind an element with an equal key, which will not be "repaired" later.

# Jupyter Notebook


## Selection Sort

Let's include the selection sort algorithm from the lecture. You can uncomment the `print` statements to get some output on its computation.

[7]:
```
def selection_sort(array):
    n = len(array)
    for i in range(n - 1):
        # print(array)
        min_index = i
        for j in range(i + 1, n):
            if array[j] < array[min_index]:
                min_index = j
        # print("Smallest element at positions", i, "-", len(array) - 1,
        #       "is", array[min_index], "at position", min_index)
        # print("Swap it with", array[i], "at position", i)
        array[i], array[min_index] = array[min_index], array[i]
        # print(array)
```

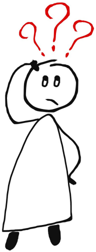
Using it in an example:

[8]:
```
test_array = [7,3,5,9,3]
selection_sort(test_array)
print(test_array)
```

```
[3, 3, 5, 7, 9]
```

Jupyter notebook: `selection_sort.ipynb`

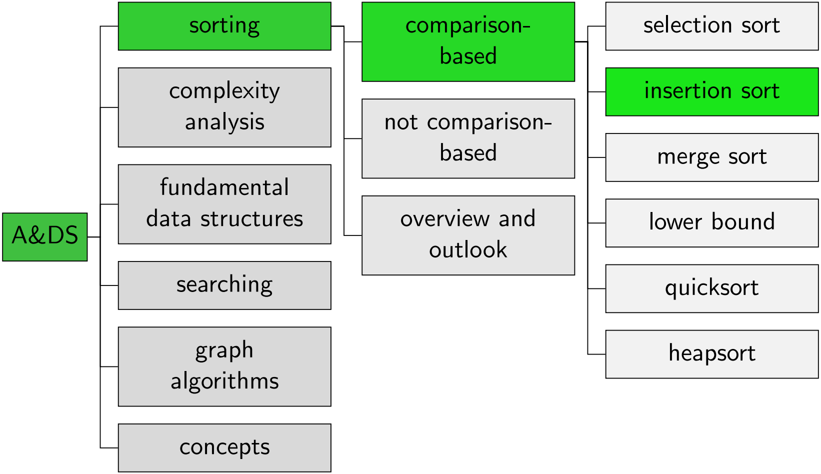
# Questions



Questions?

# Insertion Sort

# Content of the Course



# Insertion Sort: Informally

- similar to common method for sorting a hand of playing cards
- elements subsequently moved to correct position in the already sorted part of the sequence
- larger elements correspondingly moved to the right

# Insertion Sort: Example

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| | 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |
| 1 | 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |

# Insertion Sort: Example

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
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| 1 | 3 | 7 | 2 | 9 | 7 | 1 | 4 | 5 |
| 2 | 2 | 3 | 7 | 9 | 7 | 1 | 4 | 5 |

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| 3 | 2 | 3 | 7 | 9 | 7 | 1 | 4 | 5 |
| 4 | 2 | 3 | 7 | 7 | 9 | 1 | 4 | 5 |
| 5 | 1 | 2 | 3 | 7 | 7 | 9 | 4 | 5 |
| 6 | 1 | 2 | 3 | 4 | 7 | 7 | 9 | 5 |
| 7 | 1 | 2 | 3 | 4 | 5 | 7 | 7 | 9 |

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| 3 | 2 | 3 | 7 | 9 | 7 | 1 | 4 | 5 |
| 4 | 2 | 3 | 7 | 7 | 9 | 1 | 4 | 5 |
| 5 | 1 | 2 | 3 | 7 | 7 | 9 | 4 | 5 |
| 6 | 1 | 2 | 3 | 4 | 7 | 7 | 9 | 5 |
| 7 | 1 | 2 | 3 | 4 | 5 | 7 | 7 | 9 |

gray entries not moved

red entry moved into sorted range

black entries moved one position to the right

# Insertion Sort: Algorithm

```python
def insertion_sort(array):
    n = len(array)
    for i in range(1, n):  # i = 1, ..., n - 1
        # move array[i] to the left until it is
        # at the correct position.
        j = i
        while j > 0 and array[j - 1] > array[j]:
            # not yet at final position.
            # swap array[j] and array[j-1]
            array[j], array[j-1] = array[j-1], array[j]
            j -= 1
```

# Insertion Sort: Algorithm (Slightly Faster)

previous variant: most assignments to `array[j-1]` unnecessary

```
def insertion_sort(array):
    for i in range(1, len(array)):
        val = array[i]
        j = i
        while j > 0 and array[j - 1] > val:
            array[j] = array[j - 1]
            j -= 1
        array[j] = val
```

runtime analysis (later): no fundamental difference

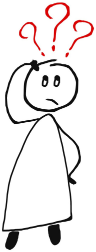
nevertheless: preferable if direct assignment possible

# Properties of Insertion Sort

- in-place: additional storage does not depend on input size
- running time: adaptive for partially sorted inputs
  - with already sorted input, immediate exit from inner loop
  - with reversely sorted input, every element moved step-by-step to the front

  exact analysis: later
- stable: elements only moved to the left as long it is swapped with a strictly larger element.
  $\rightarrow$ cannot change relative order with an equal element

# Questions



Questions?

# Summary

# Summary

- selection sort and insertion sort are two simple sorting algorithms.
- selection sort builds the sorted sequence from left to right by successively swapping a minimal element from the unsorted range to the end of the sorted range.
- insertion sort considers the elements from left to right and moves them to the correct position in the already sorted range at the beginning of the sequence.