Theory of Computer Science
E1. LOOP & WHILE Computability

Gabriele Röger

University of Basel

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Overview: Course

contents of this course:

A. background ✓
   - ▷ mathematical foundations and proof techniques

B. automata theory and formal languages ✓
   - ▷ What is a computation?

C. Turing computability ✓
   - ▷ What can be computed at all?

D. complexity theory ✓
   - ▷ What can be computed efficiently?

E. more computability theory
   - ▷ Other models of computability

E1.1 Introduction

Formal Models of Computation: LOOP/WHILE/GOTO

Formal Models of Computation

- Turing machines
- LOOP, WHILE and GOTO programs
- (primitive recursive and μ-recursive functions)

In this and the following chapter we get to know
three simple models of computation (programming languages)
and compare their power to Turing machines:

- LOOP programs $\rightsquigarrow$ today
- WHILE programs $\rightsquigarrow$ today
- GOTO programs $\rightsquigarrow$ F2
- Comparison to DTMs $\rightsquigarrow$ F2

LOOP, WHILE and GOTO Programs: Basic Concepts

- LOOP, WHILE and GOTO programs are structured like programs in (simple) "traditional" programming languages
- use finitely many variables from the set $\{x_0, x_1, x_2, \dots\}$ that can take on values in $\mathbb{N}_0$
- differ from each other in the allowed "statements"

E1.2 LOOP Programs

LOOP Programs: Syntax

Definition (LOOP Program)

LOOP programs are inductively defined as follows:

- $x_i := x_j + c$ is a LOOP program for every $i, j, c \in \mathbb{N}_0$ (addition)
- $x_i := x_j - c$ is a LOOP program for every $i, j, c \in \mathbb{N}_0$ (modified subtraction)
- If P_1 and P_2 are LOOP programs, then so is $P_1;P_2$ (composition)
- If P is a LOOP program, then so is LOOP x_i DO P END for every $i \in \mathbb{N}_0$ (LOOP loop)

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

A LOOP program **computes** a k-ary function $f : \mathbb{N}_0^k \to \mathbb{N}_0$. The computation of $f(n_1, \dots, n_k)$ works as follows:

1. Initially, the variables $x_1, \dots, x_k$ hold the values $n_1, \dots, n_k$. All other variables hold the value 0.
2. During computation, the program modifies the variables as described on the following slides.
3. The result of the computation ($f(n_1, \dots, n_k)$) is the value of x_0 after the execution of the program.

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

effect of $x_i := x_j + c$:

- The variable x_i is assigned the current value of x_j plus c.
- All other variables retain their value.

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

effect of $x_i := x_j - c$:

- The variable x_i is assigned the current value of x_j minus c if this value is non-negative.
- Otherwise x_i is assigned the value 0.
- All other variables retain their value.

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

effect of P_1; P_2:

- First, execute P_1.
  Then, execute P_2 (on the modified variable values).

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

effect of LOOP x_i DO P END:

- Let m be the value of variable x_i at the start of execution.
- The program P is executed m times in sequence.

LOOP-Computable Functions

Definition (LOOP-Computable)

A function $f : \mathbb{N}_0^k \rightarrow_{\mathrm{p}} \mathbb{N}_0$ is called LOOP-computable if a LOOP program that computes f exists.

Note: non-total functions are never LOOP-computable. (Why not?)

LOOP Programs: Example

Example (LOOP program for $f(x_1, x_2)$)

LOOP x_1 DO
  LOOP x_2 DO
    $x_0 := x_0 + 1$
  END
END

Which (binary) function does this program compute?

Syntactic Sugar or Essential Feature?

- We investigate the power of programming languages and other computation formalisms.
- Rich language features help when writing complex programs.
- Minimalistic formalisms are useful for proving statements over all programs.

$\rightsquigarrow$ conflict of interest!

Idea:

- Use minimalistic core for proofs.
- Use syntactic sugar when writing programs.

Example: Syntactic Sugar

Example (syntactic sugar)

We propose five new syntax constructs (with the obvious semantics):

- $x_i := x_j$ for $i, j \in \mathbb{N}_0$
- $x_i := c$ for $i, c \in \mathbb{N}_0$
- $x_i := x_j + x_k$ for $i, j, k \in \mathbb{N}_0$
- IF $x_i \neq 0$ THEN P END for $i \in \mathbb{N}_0$
- IF $x_i = c$ THEN P END for $i, c \in \mathbb{N}_0$

Can we simulate these with the existing constructs?

Example: Syntactic Sugar

Example (syntactic sugar)

$x_i := x_j$ for $i, j \in \mathbb{N}_0$

Simple abbreviation for $x_i := x_j + 0$.

Example: Syntactic Sugar

Example (syntactic sugar)

$x_i := c$ for $i, c \in \mathbb{N}_0$

Simple abbreviation for $x_i := x_j + c$,
where x_j is a fresh variable, i.e., an otherwise unused variable that is not an input variable.
(Thus x_j must always have the value 0 in all executions.)

Example: Syntactic Sugar

Example (syntactic sugar)

$x_i := x_j + x_k$ for $i, j, k \in \mathbb{N}_0$

Abbreviation for:

$x_i := x_j$;
LOOP x_k DO
  $x_i := x_i + 1$
END

Analogously we will also use the following:

- $x_i := x_j - x_k$
- $x_i := x_j + x_k - c - x_m + d$
- etc.

Example: Syntactic Sugar

Example (syntactic sugar)

IF $x_i \neq 0$ THEN P END for $i \in \mathbb{N}_0$

Abbreviation for:

```
x_j := 0;
LOOP x_i DO
  x_j := 1
END;
LOOP x_j DO
  P
END
```

where x_j is a fresh variable.

Example: Syntactic Sugar

Example (syntactic sugar)

IF $x_i = c$ THEN P END for $i, c \in \mathbb{N}_0$

Abbreviation for:

$x_j := 1;$
$x_k := x_i - c;$
IF $x_k \neq 0$ THEN $x_j := 0$ END;
$x_k := c - x_i;$
IF $x_k \neq 0$ THEN $x_j := 0$ END;
IF $x_j \neq 0$ THEN
  P
END

where x_j and x_k are fresh variables.

E1.3 WHILE Programs

WHILE Programs: Syntax

Definition (WHILE Program)

WHILE programs are inductively defined as follows:

- $x_i := x_j + c$ is a WHILE program for every $i, j, c \in \mathbb{N}_0$ (addition)
- $x_i := x_j - c$ is a WHILE program for every $i, j, c \in \mathbb{N}_0$ (modified subtraction)
- If P_1 and P_2 are WHILE programs, then so is $P_1;P_2$ (composition)
- If P is a WHILE program, then so is WHILE $x_i \neq 0$ DO P END for every $i \in \mathbb{N}_0$ (WHILE loop)

WHILE Programs: Semantics

Definition (Semantics of WHILE Programs)

The semantics of WHILE programs is defined exactly as for LOOP programs.

effect of WHILE $x_i \neq 0$ DO P END:

- If x_i holds the value 0, program execution finishes.
- Otherwise execute P.
- Repeat these steps until execution finishes (potentially infinitely often).

WHILE-Computable Functions

Definition (WHILE-Computable)

A function $f : \mathbb{N}_0^k \to_{\mathsf{p}} \mathbb{N}_0$ is called WHILE-computable if a WHILE program that computes f exists.

WHILE-Program: Example

Example

WHILE $x_1 \neq 0$ **DO**
$x_1 := x_1 - x_2$;
$x_0 := x_0 + 1$
END

What function $f(x_1, x_2)$ does this program compute?

E1.4 WHILE vs. LOOP

WHILE-Computability vs. LOOP-Computability

Theorem

Every LOOP-computable function is WHILE-computable.
The converse is not true.

WHILE programs are therefore strictly more powerful than LOOP programs.

WHILE-Computability vs. LOOP-Computability

Proof.

Part 1: Every LOOP-computable function is WHILE-computable.

Given any LOOP program, we construct an equivalent WHILE program, i. e., one computing the same function.

To do so, replace each occurrence of LOOP x_i DO P END with

$x_j := x_i$;
WHILE $x_j \neq 0$ DO
  $x_j := x_j - 1$;
  P
END

where x_j is a fresh variable. . . .

WHILE-Computability vs. LOOP-Computability

Proof (continued).

Part 2: Not all WHILE-computable functions are LOOP-computable.

The WHILE program

$x_1 := 1;$
WHILE $x_1 \neq 0$ DO
  $x_1 := 1$
END

computes the function $\Omega : \mathbb{N}_0 \to_{\mathrm{p}} \mathbb{N}_0$ that is undefined everywhere.

Ω is hence WHILE-computable, but not LOOP-computable (because LOOP-computable functions are always total). □

Syntactic Sugar

As we can simulate LOOP loops from LOOP programs with WHILE programs, we can use all syntactic sugar we have seen for LOOP programs in WHILE programs e.g.

- $x_i := x_j$ for $i, j \in \mathbb{N}_0$
- $x_i := c$ for $i, c \in \mathbb{N}_0$
- $x_i := x_j + x_k$ for $i, j, k \in \mathbb{N}_0$
- IF $x_i \neq 0$ THEN P END for $i \in \mathbb{N}_0$
- IF $x_i = c$ THEN P END for $i, c \in \mathbb{N}_0$

LOOP vs. WHILE: Is There a Practical Difference?

- We have shown that WHILE programs are strictly more powerful than LOOP programs.
- The example we used is not very relevant in practice because our argument only relied on the fact that LOOP-computable functions are always total.
- To terminate for every input is not much of a problem in practice. (Quite the opposite.)
- Are there any total functions that are WHILE-computable, but not LOOP-computable?

Ackermann Function: History

- David Hilbert (1926) conjectured that all computable total functions are primitive recursive (= LOOP-computable).
- Wilhelm Ackermann refuted the conjecture by supplying a counterexample (1928).
- The counterexample was simplified by Rózsa Péter (1935).

$\rightsquigarrow$ here: simplified version

Ackermann Function

Definition (Ackermann function)

The **Ackermann function** $a : \mathbb{N}_0^2 \to \mathbb{N}_0$ is defined as follows:

$$
\begin{aligned}
a(0, y) &= y + 1 && \text{for all } y \geq 0 \\
a(x, 0) &= a(x - 1, 1) && \text{for all } x > 0 \\
a(x, y) &= a(x - 1, a(x, y - 1)) && \text{for all } x, y > 0
\end{aligned}
$$

Note: the recursion in the definition is bounded,
so this defines a total function.

Table of Values

	$y = 0$	$y = 1$	$y = 2$	$y = 3$	$y = k$
$a(0, y)$	1	2	3	4	$k + 1$
$a(1, y)$	2	3	4	5	$k + 2$
$a(2, y)$	3	5	7	9	$2k + 3$
$a(3, y)$	5	13	29	61	$2^{k+3} - 3$
$a(4, y)$	13	65533	$2^{65536} - 3$	$2^{2^{65536}} - 3$	$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{k+3} - 3$

Computability of the Ackermann Function

Theorem

*The Ackermann function is WHILE-computable,
but not LOOP-computable.*

(Without proof.)

E1.5 Summary

Summary

- new models of computation for numerical functions:
  LOOP and WHILE programs
- closer to typical programming languages than Turing machines
- WHILE programs strictly more powerful than LOOP programs.
- WHILE-, but not LOOP-computable functions:
  - simple example: function that is undefined everywhere
  - more interesting example (total function):
    Ackermann function, which grows too fast
    to be LOOP-computable