Theory of Computer Science

D6. Beyond NP

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Complexity Theory: What we already have seen

- Complexity theory investigates which problems are "easy" to solve and which ones are "hard".
- two important problem classes:
  - P: problems that are solvable in polynomial time by "normal" computation mechanisms
  - NP: problems that are solvable in polynomial time with the help of nondeterminism
- We know that $\mathsf{P} \subseteq \mathsf{NP}$, but we do not know whether $\mathsf{P} = \mathsf{NP}$.
- Many practically relevant problems are NP-complete:
  - They belong to NP.
  - All problems in NP can be polynomially reduced to them.
- If there is an efficient algorithm for one NP-complete problem, then there are efficient algorithms for all problems in NP.

coNP

Complexity Class coNP

Definition (coNP)

coNP is the set of all languages L for which $\bar{L} \in \mathrm{NP}$.

Example: The complement of SAT is in coNP.

Hardness and Completeness

Definition (Hardness and Completeness)

Let C be a complexity class.

A problem Y is called C-hard if $X \leq_{\mathrm{p}} Y$ for all problems $X \in \mathrm{C}$.

Y is called C-complete if $Y \in \mathrm{C}$ and Y is C-hard.

Example (Tautology)

The following problem Tautology is coNP-complete:

Given: a propositional logic formula φ

Question: Is φ valid, i.e. is it true under all variable assignments?

Known Results and Open Questions

Open

- NP $\stackrel{?}{=}$ coNP

Known

- P $\subseteq$ coNP
- If X is NP-complete then $\bar{X}$ is coNP-complete.
- If NP $\neq$ coNP then P $\neq$ NP.
- If a coNP-complete problem is in NP, then NP $=$ coNP.
- If a coNP-complete problem is in P, then P $=$ coNP $=$ NP.

Time and Space Complexity

Reminder: Time Complexity Classes

Definition (Time Complexity Classes TIME and NTIME)

Let $t : \mathbb{N} \to \mathbb{R}^+$ be a function.

The time complexity class TIME(t(n)) is the collection of all languages that are decidable by an $O(t)$ time Turing machine, and NTIME(t(n)) is the collection of all languages that are decidable by an $O(t)$ time nondeterministic Turing machine.

- TIME(f): all languages accepted by a DTM in time f.
- NTIME(f): all languages accepted by a NTM in time f.
- $\mathrm{P} = \bigcup_{k\in\mathbb{N}} \mathrm{TIME}(n^k)$
- $\mathrm{NP} = \bigcup_{k\in\mathbb{N}} \mathrm{NTIME}(n^k)$

Space

- Analogously: A TM decides a language L in space f if the computation on every input visits at most $f(|w|)$ tape cells besides it input on the tape.
- SPACE(f): all languages decided by a DTM in space f.
- NSPACE(f): all languages decided by a NTM in space f.

Important Complexity Classes Beyond NP

- $\text{PSPACE} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(n^k)$
- $\text{NPSPACE} = \bigcup_{k \in \mathbb{N}} \text{NSPACE}(n^k)$
- $\text{EXPTIME} = \bigcup_{k \in \mathbb{N}} \text{TIME}(2^{n^k})$
- $\text{EXPSPACE} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(2^{n^k})$

Some known results:

- PSPACE = NPSPACE (from Savitch's theorem)
- PSPACE $\subseteq$ EXPTIME $\subseteq$ EXPSPACE (at least one relationship strict)
- P $\neq$ EXPTIME, PSPACE $\neq$ EXPSPACE
- P $\subseteq$ NP $\subseteq$ PSPACE

Polynomial Hierarchy

Oracle Machines

An oracle machine is like a Turing machine that has access to an oracle which can solve some decision problem in constant time.

Example oracle classes:

- $\mathsf{P}^{\mathsf{NP}} = \{L \mid L$ can get decided in polynomial time by a DTM with an oracle that decides some problem in NP$\}$
- $\mathsf{NP}^{\mathsf{NP}} = \{L \mid L$ can get decided in pol. time by a NTM with an oracle deciding some problem in NP$\}$

Polynomial Hierarchy

Inductively defined:

- $\Delta_0^P := \Sigma_0^P := \Pi_0^P := P$
- $\Delta_{i+1}^P := P^{\Sigma_i^P}$
- $\Sigma_{i+1}^P := NP^{\Sigma_i^P}$
- $\Pi_{i+1}^P := coNP^{\Sigma_i^P}$
- $PH := \bigcup_k \Sigma_k^P$

$$
\begin{tikzcd}
 & \vdots & \\
\Sigma_3^P & & \Pi_3^P \\
 & \Delta_3^P \arrow[ul] \arrow[ur] & \\
\Sigma_2^P \arrow[uu] \arrow[ur] & & \Pi_2^P \arrow[uu] \arrow[ul] \\
 & P^{NP} = \Delta_2^P \arrow[ul] \arrow[ur] & \\
NP = \Sigma_1^P \arrow[uu] \arrow[ur] & & \Pi_1^P = coNP \arrow[uu] \arrow[ul] \\
 & \Delta_0^P = \Sigma_0^P = \Pi_0^P = P = \Delta_1^P \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Polynomial Hierarchy: Results

- PH $\subseteq$ PSPACE (PH $\stackrel{?}{=}$ PSPACE is open)
- There are complete problems for each level.
- If there is a PH-complete problem, then the polynomial hierarchy collapses to some finite level.
- If P $=$ NP, the polynomial hierarchy collapses to the first level.

Counting

#P

Complexity class #P (pronounced "Sharp P")

- Set of functions $f : \{0,1\}^* \to \mathbb{N}_0$, where $f(n)$ is the number of accepting paths of a polynomial-time NTM

Example (#SAT)

The following problem $\#\mathrm{SAT}$ is #P-complete:

Given: a propositional logic formula φ

Question: Under how many variable assignments is φ true?

What's Next?

What's Next?

contents of this course:

A. background ✓
  - ▷ mathematical foundations and proof techniques

B. automata theory and formal languages ✓
  - ▷ What is a computation?

C. Turing computability ✓
  - ▷ What can be computed at all?

D. complexity theory
  - ▷ What can be computed efficiently?

E. more computability theory
  - ▷ Other models of computability

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