

Theory of Computer Science

D4. Some NP-Complete Problems, Part I

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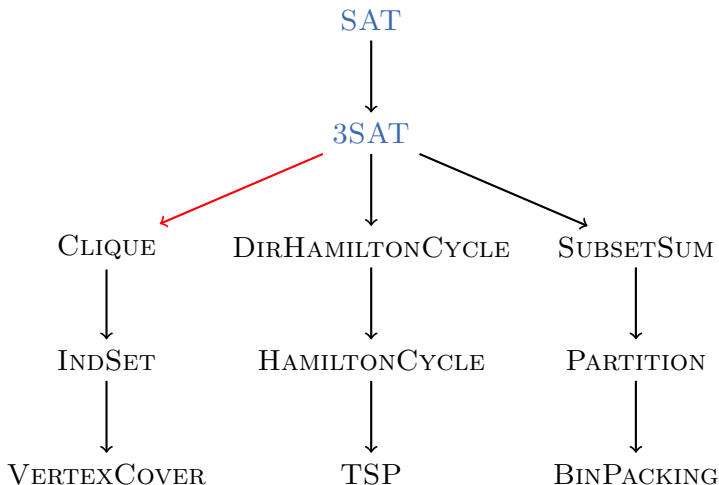
Theory of Computer Science

May 10, 2023 — D4. Some NP-Complete Problems, Part I

D4.1 Graph Problems

D4.2 Routing Problems

D4.1 Graph Problems

$3SAT \leq_p CLIQUE$ 

CLIQUE

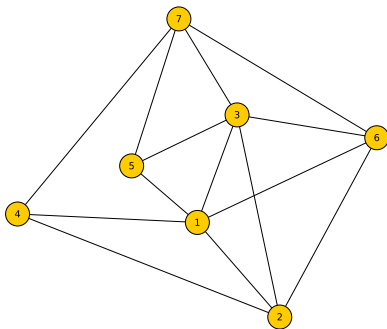
Definition (CLIQUE)

The problem **CLIQUE** is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$, number $K \in \mathbb{N}_0$

Question: Does G have a clique of size at least K ,
i. e., a set of vertices $C \subseteq V$ with $|C| \geq K$
and $\{u, v\} \in E$ for all $u, v \in C$ with $u \neq v$?

Cliques: Exercise (slido)



How many nodes has the largest clique of this graph?



CLIQUE is NP-Complete (1)

Theorem (CLIQUE is NP-Complete)

CLIQUE is NP-complete.

CLIQUE is NP-Complete (2)

Proof.

CLIQUE \in NP: guess and check.

CLIQUE is NP-hard: We show $3SAT \leq_p$ CLIQUE.

- ▶ We are given a 3-CNF formula φ , and we may assume that each clause has exactly three literals.
- ▶ In polynomial time, we must construct a graph $G = \langle V, E \rangle$ and a number K such that: G has a clique of size at least K iff φ is satisfiable.

\rightsquigarrow construction of V, E, K on the following slides.

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CLIQUE is NP-Complete (3)

Proof (continued).

Let m be the number of clauses in φ .

Let l_{ij} the j -th literal in clause i .

Define V , E , K as follows:

- ▶ $V = \{\langle i, j \rangle \mid 1 \leq i \leq m, 1 \leq j \leq 3\}$
 \rightsquigarrow a vertex for every literal of every clause
- ▶ E contains edge between $\langle i, j \rangle$ and $\langle i', j' \rangle$ if and only if
 - ▶ $i \neq i' \rightsquigarrow$ belong to **different clauses**, and
 - ▶ l_{ij} and $l_{i'j'}$ are **not complementary literals**
- ▶ $K = m$

\rightsquigarrow obviously polynomially computable

to show: reduction property

...

CLIQUE is NP-Complete (4)

Proof (continued).

(\Rightarrow): If φ is satisfiable, then $\langle V, E \rangle$ has clique of size at least K :

- ▶ Given a satisfying variable assignment choose a vertex corresponding to a satisfied literal in each clause.
- ▶ The chosen K vertices are all connected with each other and hence form a clique of size K .

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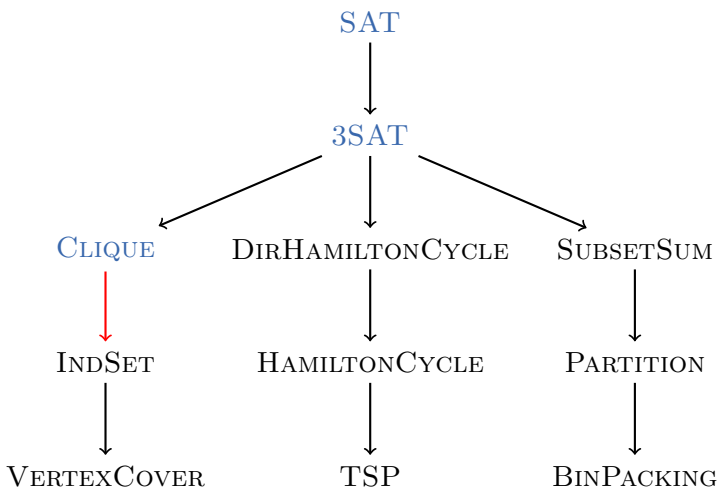
CLIQUE is NP-Complete (5)

Proof (continued).

(\Leftarrow): If $\langle V, E \rangle$ has a clique of size at least K , then φ is satisfiable:

- ▶ Consider a given clique C of size at least K .
- ▶ The vertices in C must all correspond to different clauses (vertices in the same clause are not connected by edges).
- ~> exactly one vertex per clause is included in C
- ▶ Two vertices in C never correspond to complementary literals X and $\neg X$ (due to the way we defined the edges).
- ▶ If a vertex corresp. to X was chosen, map X to T (true).
- ▶ If a vertex corresp. to $\neg X$ was chosen, map X to F (false).
- ▶ If neither was chosen, arbitrarily map X to T or F.
- ~> satisfying assignment



$\text{CLIQUE} \leq_p \text{INDSET}$ 

INDSET

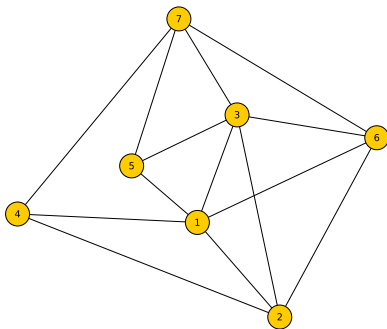
Definition (INDSET)

The problem **INDSET** is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$, number $K \in \mathbb{N}_0$

Question: Does G have an independent set of size at least K ,
i. e., a set of vertices $I \subseteq V$ with $|I| \geq K$
and $\{u, v\} \notin E$ for all $u, v \in I$ with $u \neq v$?

Independent Set: Exercise (slido)



Does this graph have an independent set of size 3?



INDSET is NP-Complete (1)

Theorem (INDSET is NP-Complete)

INDSET *is NP-complete*.

Proof.

INDSET \in NP: guess and check. ...

INDSET is NP-Complete (2)

Proof (continued).

INDSET is NP-hard: We show $\text{CLIQUE} \leq_p \text{INDSET}$.

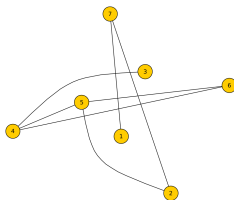
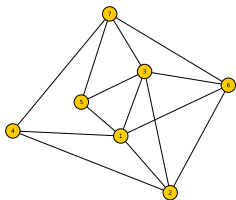
We describe a polynomial reduction f .

Let $\langle G, K \rangle$ with $G = \langle V, E \rangle$ be the given input for **CLIQUE**.

Then $f(\langle G, K \rangle)$ is the **INDSET** instance $\langle \overline{G}, K \rangle$, where $\overline{G} := \langle V, \overline{E} \rangle$ and $\overline{E} := \{\{u, v\} \subseteq V \mid u \neq v, \{u, v\} \notin E\}$.
(This graph \overline{G} is called the **complement graph** of G .)

Clearly f can be computed in polynomial time.

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INDSET is NP-Complete (3)

Proof (continued).

We have:

$$\langle \langle V, E \rangle, K \rangle \in \text{CLIQUE}$$

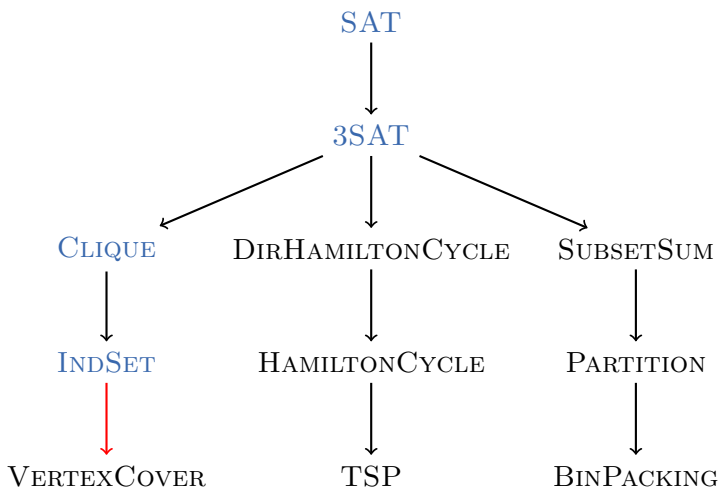
iff there exists a set $V' \subseteq V$ with $|V'| \geq K$
 and $\{u, v\} \in E$ for all $u, v \in V'$ with $u \neq v$

iff there exists a set $V' \subseteq V$ with $|V'| \geq K$
 and $\{u, v\} \notin \bar{E}$ for all $u, v \in V'$ with $u \neq v$

iff $\langle \langle V, \bar{E} \rangle, K \rangle \in \text{INDSET}$

iff $f(\langle \langle V, E \rangle, K \rangle) \in \text{INDSET}$

and hence f is a reduction. □

$\text{INDSET} \leq_p \text{VERTEXCOVER}$ 

VERTEXCOVER

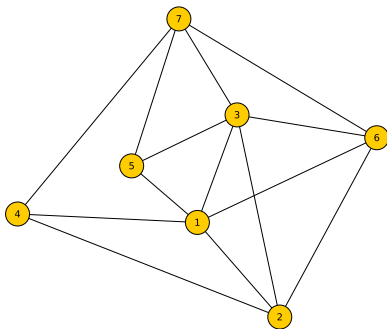
Definition (VERTEXCOVER)

The problem **VERTEXCOVER** is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$, number $K \in \mathbb{N}_0$

Question: Does G have a vertex cover of size at most K ,
i. e., a set of vertices $C \subseteq V$ with $|C| \leq K$ and $\{u, v\} \cap C \neq \emptyset$
for all $\{u, v\} \in E$?

Vertex Cover: Exercise (slido)



Does this graph have a vertex cover of size 4?



VERTEXCOVER is NP-Complete (1)

Theorem (VERTEXCOVER is NP-Complete)
VERTEXCOVER is *NP-complete*.

VERTEXCOVER is NP-Complete (2)

Proof.

VERTEXCOVER \in NP: guess and check.

VERTEXCOVER is NP-hard:

We show $\text{INDSET} \leq_p \text{VERTEXCOVER}$.

We describe a polynomial reduction f .

Let $\langle G, K \rangle$ with $G = \langle V, E \rangle$ be the given input for INDSET.

Then $f(\langle G, K \rangle) := \langle G, |V| - K \rangle$.

This can clearly be computed in polynomial time.

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VERTEXCOVER is NP-Complete (3)

Proof (continued).

For vertex set $V' \subseteq V$, we write $\overline{V'}$ for its **complement** $V \setminus V'$.

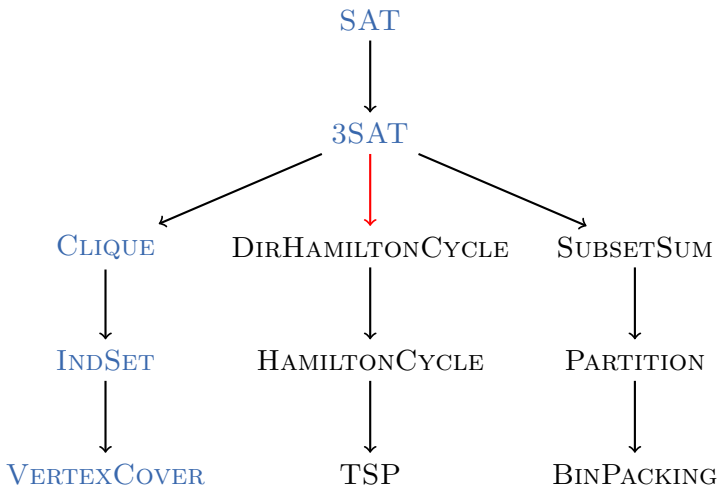
Observation: a set of vertices is a vertex cover
iff its complement is an independent set.

We thus have:

$$\begin{aligned}
 & \langle \langle V, E \rangle, K \rangle \in \text{INDSET} \\
 \text{iff} & \langle V, E \rangle \text{ has an independent set } I \text{ with } |I| \geq K \\
 \text{iff} & \langle V, E \rangle \text{ has a vertex cover } C \text{ with } |\overline{C}| \geq K \\
 \text{iff} & \langle V, E \rangle \text{ has a vertex cover } C \text{ with } |C| \leq |V| - K \\
 \text{iff} & \langle \langle V, E \rangle, |V| - K \rangle \in \text{VERTEXCOVER} \\
 \text{iff} & f(\langle \langle V, E \rangle, K \rangle) \in \text{VERTEXCOVER}
 \end{aligned}$$

and hence f is a reduction. □

D4.2 Routing Problems

$3SAT \leq_p \text{DIRHAMILTONCYCLE}$ 

DIRHAMILTONCYCLE is NP-Complete (1)

Definition (Reminder: DIRHAMILTONCYCLE)

The problem **DIRHAMILTONCYCLE** is defined as follows:

Given: directed graph $G = \langle V, E \rangle$

Question: Does G contain a Hamilton cycle?

Theorem

DIRHAMILTONCYCLE *is NP-complete.*

DIRHAMILTONCYCLE is NP-Complete (2)

Proof.

DIRHAMILTONCYCLE \in NP: guess and check.

DIRHAMILTONCYCLE is NP-hard:

We show $3SAT \leq_p$ DIRHAMILTONCYCLE.

- ▶ We are given a 3-CNF formula φ where each clause contains exactly three literals and no clause contains duplicated literals.
- ▶ We must, in polynomial time, construct a directed graph $G = \langle V, E \rangle$ such that:
 G contains a Hamilton cycle iff φ is satisfiable.
- ▶ construction of $\langle V, E \rangle$ on the following slides

...

DIRHAMILTONCYCLE is NP-Complete (3)

Proof (continued).

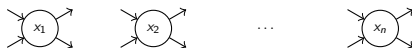
- ▶ Let X_1, \dots, X_n be the atomic propositions in φ .
- ▶ Let c_1, \dots, c_m be the clauses of φ with $c_i = (l_{i1} \vee l_{i2} \vee l_{i3})$.
- ▶ Construct a graph with $6m + n$ vertices (described on the following slides).

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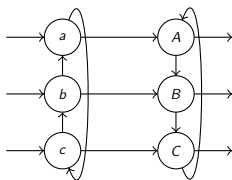
DIRHAMILTONCYCLE is NP-Complete (4)

Proof (continued).

- ▶ For every variable X_i , add vertex x_i with 2 incoming and 2 outgoing edges:



- ▶ For every clause c_j , add the subgraph C_j with 6 vertices:



- ▶ We describe later how to connect these parts.

...

DIRHAMILTONCYCLE is NP-Complete (5)

Proof (continued).

Let π be a Hamilton cycle of the total graph.

- ▶ Whenever π enters subgraph C_j from one of its “entrances”, it must leave via the corresponding “exit”:
($a \rightarrow A, b \rightarrow B, c \rightarrow C$).

Otherwise, π cannot be a Hamilton cycle.

- ▶ Hamilton cycles can behave in the following ways with regard to C_j :
 - ▶ π passes through C_j once (from any entrance)
 - ▶ π passes through C_j twice (from any two entrances)
 - ▶ π passes through C_j three times (once from every entrance)

...

DIRHAMILTONCYCLE is NP-Complete (6)

Proof (continued).

Connect the “open ends” in the graph as follows:

- ▶ Identify entrances/exits of the clause subgraph C_j with the three literals in clause c_j .
- ▶ One exit of x_i is **positive**, the other one is **negative**.
- ▶ For the **positive** exit, determine the clauses in which the positive literal X_i occurs:
 - ▶ Connect the positive exit of x_i with the X_i -entrance of the first such clause graph.
 - ▶ Connect the X_i -exit of this clause graph with the X_i -entrance of the second such clause graph, and so on.
 - ▶ Connect the X_i -exit of the last such clause graph with the positive entrance of x_{i+1} (or x_1 if $i = n$).
- ▶ analogously for the **negative** exit of x_i and the literal $\neg X_i$

...

DIRHAMILTONCYCLE is NP-Complete (7)

Proof (continued).

The construction is polynomial and is a reduction:

(\Rightarrow): **construct a Hamilton cycle from a satisfying assignment**

- ▶ Given a satisfying assignment \mathcal{I} , construct a Hamilton cycle that leaves x_i through the positive exit if $\mathcal{I}(X_i)$ is true and by the negative exit if $\mathcal{I}(X_i)$ is false.
- ▶ Afterwards, we visit all C_j -subgraphs for clauses that are satisfied by this literal.
- ▶ In total, we visit each C_j -subgraph 1–3 times.

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DIRHAMILTONCYCLE is NP-Complete (8)

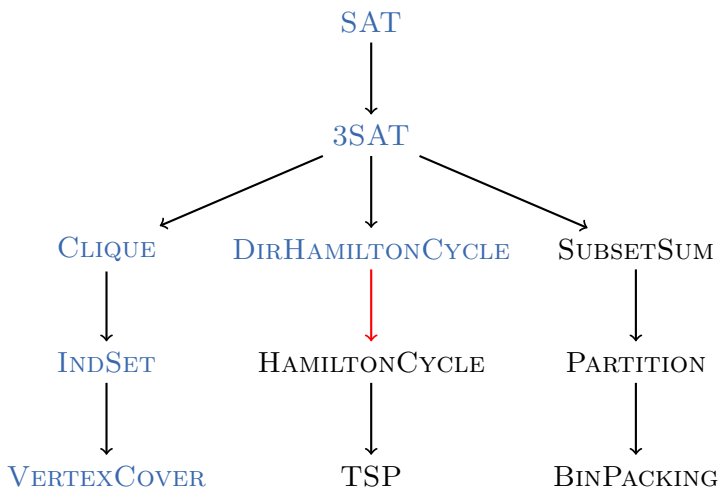
Proof (continued).

(\Leftarrow): **construct a satisfying assignment from a Hamilton cycle**

- ▶ A Hamilton cycle visits every vertex x_i and leaves it by the positive or negative exit.
- ▶ Map X_i to true or false depending on which exit is used to leave x_i .
- ▶ Because the cycle must traverse each C_j -subgraph at least once (otherwise it is not a Hamilton cycle), this results in a satisfying assignment. (Details omitted.)



DIRHAMILTONCYCLE \leq_p HAMILTONCYCLE



HAMILTONCYCLE is NP-Complete (1)

Definition (Reminder: HAMILTONCYCLE)

The problem **HAMILTONCYCLE** is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$

Question: Does G contain a Hamilton cycle?

Theorem

HAMILTONCYCLE is *NP-complete*.

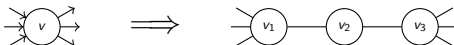
HAMILTONCYCLE is NP-Complete (2)

Proof sketch.

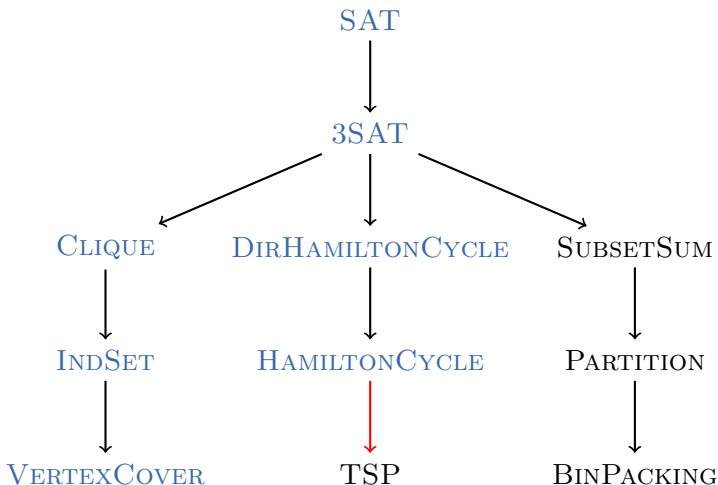
HAMILTONCYCLE \in NP: guess and check.

HAMILTONCYCLE is NP-hard: We show
DIRHAMILTONCYCLE \leq_p HAMILTONCYCLE.

Basic building block of the reduction:



HAMILTONCYCLE \leq_p TSP



TSP is NP-Complete (1)

Definition (Reminder: TSP)

TSP (traveling salesperson problem) is the following decision problem:

- ▶ **Given:** finite set $S \neq \emptyset$ of cities, symmetric cost function $cost : S \times S \rightarrow \mathbb{N}_0$, cost bound $K \in \mathbb{N}_0$
- ▶ **Question:** Is there a tour with total cost at most K , i. e., a permutation $\langle s_1, \dots, s_n \rangle$ of the cities with
$$\sum_{i=1}^{n-1} cost(s_i, s_{i+1}) + cost(s_n, s_1) \leq K?$$

Theorem

TSP is NP-complete.

TSP is NP-Complete (2)

Proof.

TSP \in NP: guess and check.

TSP is NP-hard: We showed $\text{HAMILTONCYCLE} \leq_p \text{TSP}$
in Chapter D2. □

Summary

- ▶ In this chapter we showed NP-completeness of
 - ▶ three classical graph problems:
CLIQUE, INDSET, VERTEXCOVER
 - ▶ three classical routing problems:
DIRHAMILTONCYCLE, HAMILTONCYCLE, TSP