Theory of Computer Science

D2. Polynomial Reductions and NP-completeness

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D2.1 Polynomial Reductions

Polynomial Reductions: Idea

- Reductions are a common and powerful concept in computer science. We know them from Part C.
- The basic idea is that we solve a new problem by reducing it to a known problem.
- In complexity theory we want to use reductions that allow us to prove statements of the following kind:
  Problem A can be solved efficiently
  if problem B can be solved efficiently.
- For this, we need a reduction from A to B that can be computed efficiently itself (otherwise it would be useless for efficiently solving A).

Polynomial Reductions

Definition (Polynomial Reduction)

Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be decision problems.
We say that A can be polynomially reduced to B,
written $A \leq_{\mathrm{p}} B$, if there is a function $f : \Sigma^* \to \Gamma^*$ such that:

- f can be computed in polynomial time by a DTM
  - i. e., there is a polynomial p and a DTM M such that M computes $f(w)$ in at most $p(|w|)$ steps given input $w \in \Sigma^*$
- f reduces A to B
  - i. e., for all $w \in \Sigma^*$: $w \in A$ iff $f(w) \in B$

f is called a polynomial reduction from A to B

Polynomial Reductions: Remarks

- Polynomial reductions are also called Karp reductions (after Richard Karp, who wrote a famous paper describing many such reductions in 1972).
- In practice, of course we do not have to specify a DTM for f: it just has to be clear that f can be computed in polynomial time by a deterministic algorithm.

Polynomial Reductions: Example (1)

Definition (HamiltonCycle)

HamiltonCycle is the following decision problem:

- Given: undirected graph $G = \langle V, E \rangle$
- Question: Does G contain a Hamilton cycle?

Reminder:

Definition (Hamilton Cycle)

A Hamilton cycle of G is a sequence of vertices in V, $\pi = \langle v_0, \dots, v_n \rangle$, with the following properties:

- π is a path: there is an edge from v_i to v_{i+1} for all $0 \leq i < n$
- π is a cycle: $v_0 = v_n$
- π is simple: $v_i \neq v_j$ for all $i \neq j$ with $i, j < n$
- π is Hamiltonian: all nodes of V are included in π

Polynomial Reductions: Example (2)

Definition (TSP)

TSP (traveling salesperson problem) is the following decision problem:

- Given: finite set $S \neq \emptyset$ of cities, symmetric cost function $cost : S \times S \to \mathbb{N}_0$, cost bound $K \in \mathbb{N}_0$
- Question: Is there a tour with total cost at most K, i. e., a permutation $\langle s_1, \dots, s_n \rangle$ of the cities with $\sum_{i=1}^{n-1} cost(s_i, s_{i+1}) + cost(s_n, s_1) \leq K$?

Polynomial Reductions: Example (3)

Theorem (HamiltonCycle $\leq_p$ TSP)

HamiltonCycle $\leq_p$ TSP.

Proof.

$\rightsquigarrow$ blackboard $\square$

Exercise: Polynomial Reduction

Definition (HamiltonianCompletion)

HamiltonianCompletion is the following decision problem:

- Given: undirected graph $G = \langle V, E \rangle$, number $k \in \mathbb{N}_0$
- Question: Can G be extended with at most k edges such that the resulting graph has a Hamilton cycle?

Show that
HamiltonCycle $\leq_p$ HamiltonianCompletion.

Reminder: P and NP

P: class of languages that are decidable in polynomial time by a deterministic Turing machine

NP: class of languages that are decidable in polynomial time by a non-deterministic Turing machine

Properties of Polynomial Reductions (1)

Theorem (Properties of Polynomial Reductions)

Let A, B and C decision problems.

1. *If $A \leq_{\mathrm{p}} B$ and $B \in P$, then $A \in P$.*
2. *If $A \leq_{\mathrm{p}} B$ and $B \in NP$, then $A \in NP$.*
3. *If $A \leq_{\mathrm{p}} B$ and $A \notin P$, then $B \notin P$.*
4. *If $A \leq_{\mathrm{p}} B$ and $A \notin NP$, then $B \notin NP$.*
5. *If $A \leq_{\mathrm{p}} B$ and $B \leq_{\mathrm{p}} C$, then $A \leq_{\mathrm{p}} C$.*

Properties of Polynomial Reductions (2)

Proof.
for 1.:

We must show that there is a DTM deciding A in polynomial time.

We know:

- There is a DTM M_B that decides B in time p, where p is a polynomial.
- There is a DTM M_f that computes a reduction from A to B in time q, where q is a polynomial.

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Properties of Polynomial Reductions (3)

Proof (continued).

Consider the machine M that first behaves like M_f, and then (after M_f stops) behaves like M_B on the output of M_f.

M decides A:

- M behaves on input w as M_B does on input $f(w)$, so it accepts w if and only if $f(w) \in B$.
- Because f is a reduction, $w \in A$ iff $f(w) \in B$.

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Properties of Polynomial Reductions (4)

Proof (continued).

Computation time of M on input w:

- first M_f runs on input w: $\leq q(|w|)$ steps
- then M_B runs on input $f(w)$: $\leq p(|f(w)|)$ steps
- $|f(w)| \leq |w| + q(|w|)$ because in $q(|w|)$ steps, M_f can write at most $q(|w|)$ additional symbols onto the tape

$\rightsquigarrow$ total computation time $\leq q(|w|) + p(|f(w)|)$
$\leq q(|w|) + p(|w| + q(|w|))$

$\rightsquigarrow$ this is polynomial in $|w|$ $\rightsquigarrow$ $A \in \mathrm{P}$.

...

Properties of Polynomial Reductions (5)

Proof (continued).

for 2.:

analogous to 1., only that M_B and M are NTMs

of 3.+4.:

equivalent formulations of 1.+2. (contraposition)

of 5.:

Let $A \leq_{\mathrm{p}} B$ with reduction f and $B \leq_{\mathrm{p}} C$ with reduction g.
Then $g \circ f$ is a reduction of A to C.

The computation time of the two computations in sequence is polynomial by the same argument used in the proof for 1. □

D2.2 NP-Hardness and NP-Completeness

NP-Hardness and NP-Completeness

Definition (NP-Hard, NP-Complete)

Let B be a decision problem.

B is called NP-hard if $A \leq_p B$ for all problems $A \in \mathrm{NP}$.

B is called NP-complete if $B \in \mathrm{NP}$ and B is NP-hard.

NP-Complete Problems: Meaning

- NP-hard problems are "at least as difficult" as all problems in NP.
- NP-complete problems are "the most difficult" problems in NP: all problems in NP can be reduced to them.
- If $A \in \mathrm{P}$ for any NP-complete problem A, then $\mathrm{P} = \mathrm{NP}$. (Why?)
- That means that either there are efficient algorithms for all NP-complete problems or for none of them.
- Do NP-complete problems actually exist?

D2.3 Summary

Summary

- polynomial reductions: $A \leq_{\mathrm{p}} B$ if there is a total function f computable in polynomial time, such that for all words w: $w \in A$ iff $f(w) \in B$
- $A \leq_{\mathrm{p}} B$ implies that A is "at most as difficult" as B
- polynomial reductions are transitive
- NP-hard problems B: $A \leq_{\mathrm{p}} B$ for all $A \in \mathrm{NP}$
- NP-complete problems B: $B \in \mathrm{NP}$ and B is NP-hard