

Theory of Computer Science

C5. Post Correspondence Problem

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Post Correspondence Problem

More Options for Reduction Proofs?

- We can prove the undecidability of a problem with a reduction from an undecidable problem.
- The **halting problem** and the **halting problem on the empty tape** are possible options for this.
- both halting problem variants are quite similar 😞

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→ Is there some problem that is different in flavor?

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Post correspondence problem

(named after mathematician **Emil Leon Post**)

Post Correspondence Problem: Example

Example (Post Correspondence Problem)

Given: different kinds of “dominos”

$$1: \begin{array}{|c|} \hline 1 \\ \hline 101 \\ \hline \end{array} \quad 2: \begin{array}{|c|} \hline 10 \\ \hline 00 \\ \hline \end{array} \quad 3: \begin{array}{|c|} \hline 011 \\ \hline 11 \\ \hline \end{array}$$

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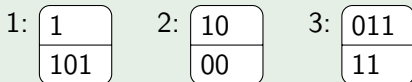
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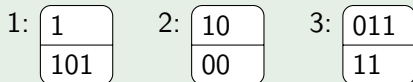
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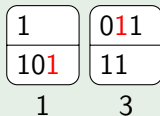
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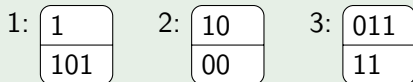
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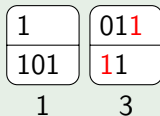
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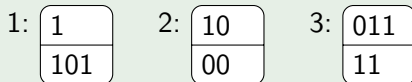
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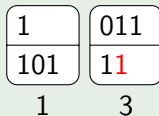
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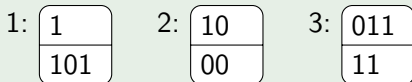
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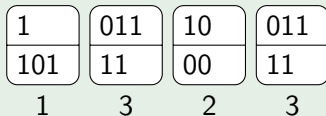
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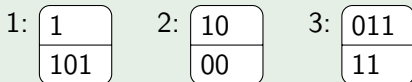
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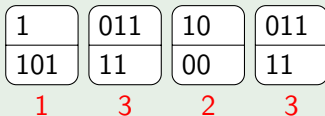
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Post Correspondence Problem: Definition

Definition (Post Correspondence Problem PCP)

Given: Finite **sequence of pairs of words**
 $(t_1, b_1), (t_2, b_2), \dots, (t_k, b_k)$, where $t_i, b_i \in \Sigma^+$
(for an arbitrary alphabet Σ)

Question: Is there a sequence
 $i_1, i_2, \dots, i_n \in \{1, \dots, k\}$, $n \geq 1$,
with $t_{i_1} t_{i_2} \dots t_{i_n} = b_{i_1} b_{i_2} \dots b_{i_n}$?

A **solution** of the correspondence problem is such a sequence i_1, \dots, i_n , which we call a **match**.

Exercise (slido)

Consider PCP instance $(11, 1), (0, 00), (10, 01), (01, 11)$.

Is 2, 4, 3, 3, 1 a match?



Given-Question Form vs. Definition as Set

So far: problems defined as sets

Now: definition in **Given-Question form**

Definition (new problem P)

Given: Instance \mathcal{I}

Question: Does \mathcal{I} have a specific property?

Given-Question Form vs. Definition as Set

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Now: definition in **Given-Question form**

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Question: Does \mathcal{I} have a specific property?

corresponds to definitions

Definition (new problem P)

The problem P is the language

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$P = \{\langle\langle\mathcal{I}\rangle\rangle \mid \mathcal{I} \text{ is an instance with the required property}\}.$

PCP Definition as Set

We can alternatively define PCP as follows:

Definition (Post Correspondence Problem PCP)

The Post Correspondence Problem PCP is the set

$$\text{PCP} = \{w \mid w \text{ encodes a sequence of pairs of words } (t_1, b_1), (t_2, b_2), \dots, (t_k, b_k), \text{ for which there is a sequence } i_1, i_2, \dots, i_n \in \{1, \dots, k\} \text{ such that } t_{i_1} t_{i_2} \dots t_{i_n} = b_{i_1} b_{i_2} \dots b_{i_n}\}.$$

Questions



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(Un-)Decidability of PCP

Post Correspondence Problem

PCP cannot be so hard, huh?

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| | | |
|------|------|-----|
| 1101 | 0110 | 1 |
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Formally: $K = ((1101, 1), (0110, 11), (1, 110))$

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→ Shortest match has length 252!

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|----|-----|-----|
| 10 | 0 | 100 |
| 0 | 001 | 1 |

Formally: $K = ((10, 0), (0, 001), (100, 1))$

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→ Unsolvable

PCP: Turing-recognizability

Theorem (Turing-recognizability of PCP)

PCP *is Turing-recognizable*.

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Theorem (Turing-recognizability of PCP)

PCP is *Turing-recognizable*.

Proof.

Recognition procedure for input w :

- If w encodes a sequence $(t_1, b_1), \dots, (t_k, b_k)$ of pairs of words:
Test systematically longer and longer sequences i_1, i_2, \dots, i_n whether they represent a match.
If yes, terminate and return “yes”.
- If w does not encode such a sequence: enter an infinite loop.

If $w \in \text{PCP}$ then the procedure terminates with “yes”, otherwise it does not terminate. □

PCP: Undecidability

Theorem (Undecidability of PCP)

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Proof via an intermediate other problem

modified PCP (MPCP)

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→ Let's get started...

MPCP: Definition

Definition (Modified Post Correspondence Problem MPCP)

Given: Sequence of word pairs as for PCP

Question: Is there a match $i_1, i_2, \dots, i_n \in \{1, \dots, k\}$
with $i_1 = 1$?

Reducibility of MPCP to PCP(1)

Lemma

MPCP \leq PCP.

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Proof.

Let $\#, \$ \notin \Sigma$. For word $w = a_1 a_2 \dots a_m \in \Sigma^+$ define

$$\bar{w} = \#a_1\#a_2\#\dots\#a_m\#$$

$$\dot{w} = \#a_1\#a_2\#\dots\#a_m$$

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For input $C = ((t_1, b_1), \dots, (t_k, b_k))$ define

$$f(C) = ((\bar{t}_1, \dot{b}_1), (\acute{t}_1, \dot{b}_1), (\acute{t}_2, \dot{b}_2), \dots, (\acute{t}_k, \dot{b}_k), (\$, \# \$))$$

Reducibility of MPCP to PCP(2)

Proof (continued).

$$f(C) = ((\bar{t}_1, \bar{b}_1), (t'_1, b_1), (t'_2, b_2), \dots, (t'_k, b_k), (\$, \#\$))$$

Function f is **computable**, and can suitably get extended to a **total** function. It holds that

C has a solution with $i_1 = 1$ iff $f(C)$ has a solution:



Reducibility of MPCP to PCP(2)

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Let $1, i_2, i_3, \dots, i_n$ be a solution for C . Then $1, i_2 + 1, \dots, i_n + 1, k + 2$ is a solution for $f(C)$.



Reducibility of MPCP to PCP(2)

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If i_1, \dots, i_n is a match for $f(C)$, then (due to the construction of the word pairs) there is a $m \leq n$ such that $i_1 = 1, i_m = k + 2$ and $i_j \in \{2, \dots, k + 1\}$ for $j \in \{2, \dots, m - 1\}$. Then $1, i_2 - 1, \dots, i_{m-1} - 1$ is a solution for C .



Reducibility of MPCP to PCP(2)

Proof (continued).

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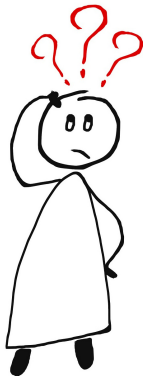
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$\Rightarrow f$ is a reduction from MPCP to PCP.



Questions



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PCP: Undecidability – Where are we?

Theorem (Undecidability of PCP)

PCP *is undecidable*.

Proof via an intermediate other problem

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Reducibility of H to MPCP(1)

Lemma

$H \leq \text{MPCP}$.

Proof.

Goal: Construct for Turing machine

$M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ and word $w \in \Sigma^*$ an MPCP instance $C = ((t_1, b_1), \dots, (t_k, b_k))$ such that

M started on w terminates iff $C \in \text{MPCP}$.

...

Reducibility of H to MPCP(2)

Proof (continued).

Idea:

- Sequence of words describes sequence of configurations of the TM

■ “ t -row” follows “ b -row” x : $\boxed{\# \ c_0 \ \# \ c_1 \ \# \ c_2 \ \#}$

y : $\boxed{\# \ c_0 \ \# \ c_1 \ \# \ c_2 \ \# \ c_3 \ \#}$

- Configurations get mostly just copied, only the area around the head changes.
- After a terminating configuration has been reached: make row equal by deleting the configuration.

...

Reducibility of H to MPCP(3)

Proof (continued).

Alphabet of C is $\Gamma \cup Q \cup \{\#\}$.

1. Pair: $(\#, \#q_0w\#)$

Other pairs:

- ① copy: (a, a) for all $a \in \Gamma \cup \{\#\}$
- ② transition:

$$(qa, cq') \text{ if } \delta(q, a) = (q', c, R)$$
$$(q\#, cq'\#) \text{ if } \delta(q, \square) = (q', c, R)$$

Reducibility of H to MPCP(4)

Proof (continued).

$(bqa, q'bc)$ if $\delta(q, a) = (q', c, L)$ for all $b \in \Gamma$

$(bq\#, q'bc\#)$ if $\delta(q, \square) = (q', c, L)$ for all $b \in \Gamma$

$(\#qa, \#q'c)$ if $\delta(q, a) = (q', c, L)$

$(\#q\#, \#q'c\#)$ if $\delta(q, \square) = (q', c, L)$

- ③ deletion: (aq, q) and (qa, q)
for all $a \in \Gamma$ and $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$
- ④ finish: $(q\#\#\#, \#)$ for all $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$

...

Reducibility of H to MPCP(5)

Proof (continued).

“ \Rightarrow ” If M terminates on input w , there is a sequence c_0, \dots, c_t of configurations with

- $c_0 = q_0 w$ is the start configuration
- c_t is a terminating configuration
($c_t = uq v$ mit $u, v \in \Gamma^*$ and $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$)
- $c_i \vdash c_{i+1}$ for $i = 0, 1, \dots, t - 1$

Reducibility of H to MPCP(5)

Proof (continued).

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Then C has a match with the overall word

$$\#c_0\#c_1\#\dots\#c_t\#c'_t\#c''_t\#\dots\#q_e\#\#$$

Up to c_t : "' t -row'" follows "' b -row'"

From c'_t : deletion of symbols adjacent to terminating state. ...

Reducibility of H to MPCP(6)

Proof (continued).

“ \Leftarrow ” If C has a solution, it has the form

$$\#c_0\#c_1\#\dots\#c_n\#\#,$$

with $c_0 = q_0w$. Moreover, there is an $\ell \leq n$, such that q_{accept} or q_{reject} occurs for the first time in c_ℓ .

All c_i for $i \leq \ell$ are configurations of M and $c_i \vdash c_{i+1}$ for $i \in \{0, \dots, \ell - 1\}$.

c_0, \dots, c_ℓ is hence the sequence of configurations of M on input w , which shows that the TM terminates. □

PCP: Undecidability – Done!

Theorem (Undecidability of PCP)

PCP *is undecidable*.

Proof via an intermediate other problem

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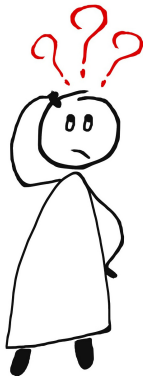
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Proof.

Due to $H \leq \text{MPCP}$ and $\text{MPCP} \leq \text{PCP}$ it holds that $H \leq \text{PCP}$.
Since H is undecidable, also PCP must be undecidable. □

Questions



Questions?

Summary

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- **Post Correspondence Problem:**
Find a sequence of word pairs s.t. the concatenation of all first components equals the one of all second components.
- The Post Correspondence Problem is **Turing-recognizable** but **not decidable**.