Theory of Computer Science
B10. Turing Machines I

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March 29, 2023

B10.1 Turing Machines

B10.2 Summary

B10.1 Turing Machines

Automata for Type-1 and Type-0 Languages?

Finite automata recognize exactly the regular languages, push-down automata exactly the context-free languages. Are there automata models for context-sensitive and type-0 languages?
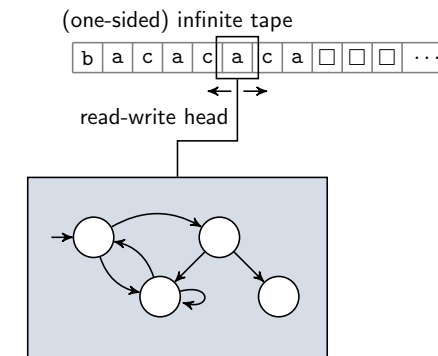
Yes! ⇝ Turing machines

Alan Turing (1912–1954)

Picture courtesy of Jon Callas /
wikimedia commons

- ▶ British logician, mathematician, cryptanalyst and computer scientist
- ▶ most important work (for us): On Computable Numbers, with an Application to the Entscheidungsproblem ⇝ Turing machines
- ▶ collaboration on Enigma decryption
- ▶ conviction due to homosexuality; pardoned by Elizabeth II in Dec. 2013
- ▶ Turing award most important science award in computer science

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Turing Machines: Conceptually

(one-sided) infinite tape

read-write head

Turing Machine: Definition

Definition (Deterministic Turing Machine)

A (deterministic) Turing machine (DTM) is given by a 7-tuple
$M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject} \rangle$, where
Q, Σ, Γ are all finite sets and

- ▶ Q is the set of states,
- ▶ Σ is the input alphabet, not containing the blank symbol $\square$,
- ▶ Γ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$,
- ▶ $\delta : (Q \setminus \{q_{accept}, q_{reject}\}) \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function,
- ▶ $q_0 \in Q$ is the start state,
- ▶ $q_{accept} \in Q$ is the accept state,
- ▶ $q_{reject} \in Q$ is the reject state, where $q_{accept} \neq q_{reject}$.


Turing Machine: Transition Function

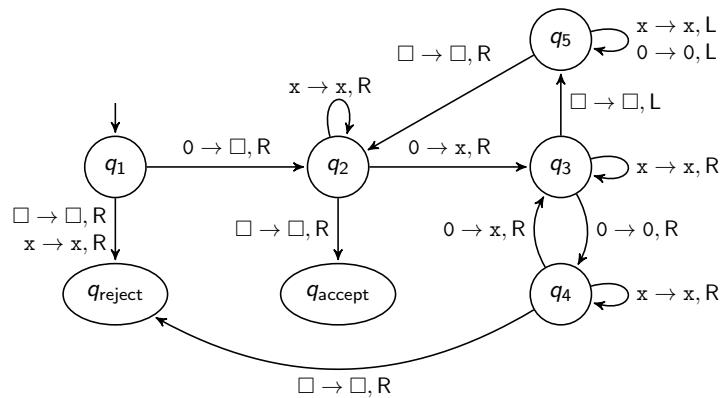Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject} \rangle$ be a DTM.

What is the Intuitive Meaning of the Transition Function δ?
$\delta(q, a) = \langle q', b, D \rangle$:

- ▶ If M is in state q and reads a, then
- ▶ M transitions to state q' in the next step,
- ▶ replacing a with b,
- ▶ and moving the head in direction $D \in \{L, R\}$, where:
  - ▶ R: one step to the right,
  - ▶ L: one step to the left, except if the head is on the left-most cell of the tape in which case there is no movement

$$q \xrightarrow{a \to b, D} q'$$
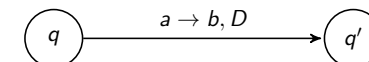
Deterministic Turing Machine: Example

$$\langle \{q_1, \ldots, q_5, q_{\text{accept}}, q_{\text{reject}}\}, \{0\}, \{0, x, \square\}, \delta, q_1, q_{\text{accept}}, q_{\text{reject}}\rangle$$

Turing Machine: Configuration

> **Definition (Configuration of a Turing Machine)**
> A configuration of a Turing machine
> $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}}\rangle$
> is given by a triple $c \in \Gamma^* \times Q \times \Gamma^+$.

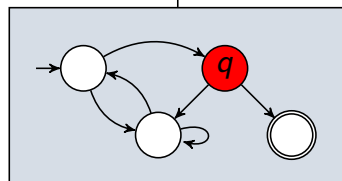Configuration $\langle w_1, q, w_2 \rangle$ intuitively means that

▶ the non-empty or already visited part of the tape contains the word $w_1 w_2$,

▶ the read-write head is on the first symbol of w_2, and

▶ the TM is in state q.

Turing Machine Configurations: Example

Example

configuration $\langle \texttt{BEFORE}, q, \texttt{AFTER}\square\square \rangle$.

Turing Machine Configurations: Start Configuration

Initially

▶ the TM is in start state q_0,

▶ the head is on the first tape cell, and

▶ the tape contains the input word w followed by an infinite number of $\square$ entries.

The corresponding start configuration is $\langle \varepsilon, q_0, w \rangle$ if $w \neq \varepsilon$ and $\langle \varepsilon, q_0, \square \rangle$ if $w = \varepsilon$.
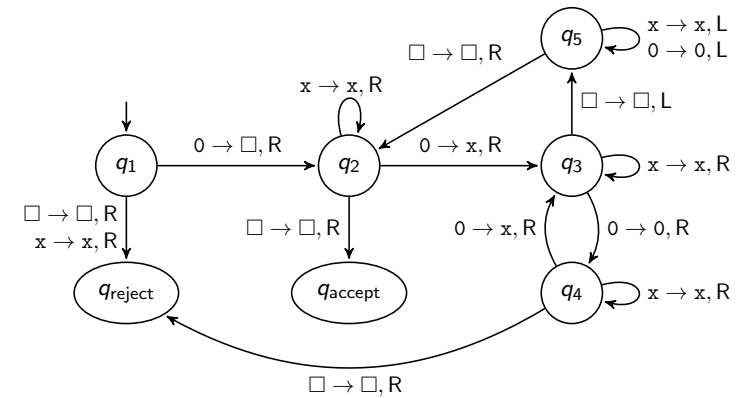
Turing Machine: Step

Definition (Transition/Step of a Turing Machine)

A DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ transitions
from configuration c to configuration c' in one step ($c \vdash_M c'$)
according to the following rules:

- $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_m c, q', b_2 \dots b_n \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathsf{R} \rangle$, $m \geq 0$, $n \geq 2$
- $\langle a_1 \dots a_m, q, b_1 \rangle \vdash_M \langle a_1 \dots a_m c, q', \square \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathsf{R} \rangle$, $m \geq 0$
- $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_{m-1}, q', a_m c b_2 \dots b_n \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathsf{L} \rangle$, $m \geq 1$, $n \geq 1$
- $\langle \varepsilon, q, b_1 \dots b_n \rangle \vdash_M \langle \varepsilon, q', c b_2 \dots b_n \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathsf{L} \rangle$, $n \geq 1$

Step: Exercise (Slido)

$\langle \square \mathsf{x}, q_3, 00 \rangle \vdash \; ?$

DTM: Accepted Words

Intuitively, a DTM accepts a word if its computation terminates in
the accept state.

Definition (Words Accepted by a DTM)

DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ accepts the word
$w = a_1 \dots a_n$ if there is a sequence of configurations $c_0, \dots, c_k$ with

1. c_0 is the start configuration of M on input w,
2. $c_i \vdash_M c_{i+1}$ for all $i \in \{0, \dots, k-1\}$, and
3. c_k is an accepting configuration,
   i.e., a configuration with state q_{accept}.

Accepted Word: Example (Slido)

Does this Turing machine accept input 0000?
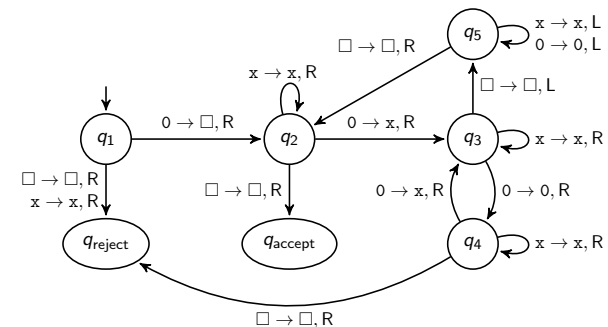
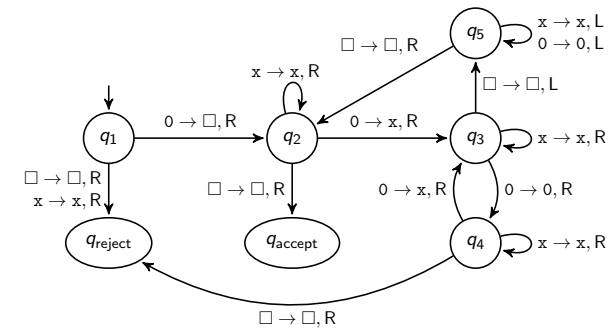## DTM: Recognized Language

Definition (Language Recognized by a DTM)

Let M be a deterministic Turing Machine

The language recognized by M (or the language of M) is defined as $\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.
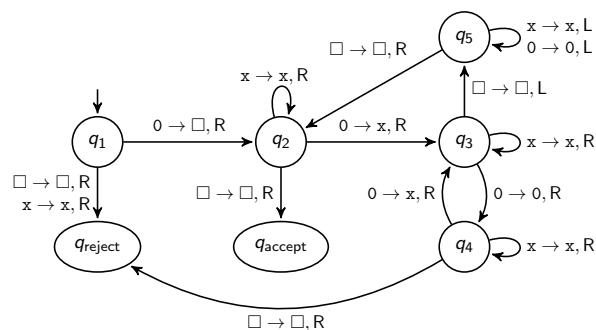
Definition (Turing-recognizable Language)

We call a language Turing-recognizable if some deterministic Turing machine recognizes it.

Turing Machine: Example

1. Sweep left to right across the tape, crossing off every other 0.
2. If in stage 1 the tape contained a single 0, accept.
3. If in stage 1 the tape contained more than one 0 and the number of 0s was odd, reject.
4. Return the head to the left end of the tape and go to stage 1.

Recognized Language: Example

What language does the Turing machine recognize?

Deciders

- A Turing machine either fails to accept an input
    - because it rejects it (entering q_{reject}) or
    - because it loops (= does not halt).
- A Turing machine that halts on all inputs (entering q_{reject} or q_{accept}) is called a decider.
- A decider that recognizes some language also is said to decide the language.

Definition (Turing-decidable Language)

We call a language Turing-decidable (or decidable) if some deterministic Turing machine decides it.

Exercise (if time)

Specify the state diagram of a DTM that decides language

$$L = \{w\#w \mid w \in \{0,1\}^*\}.$$

Feel free to solve this together with your neighbour.

B10.2 Summary

Summary

▶ Turing machines only have finitely many states but an unbounded tape as "memory".
▶ Alan Turing proposed them as a mathematical model for arbitrary algorithmic computations.
▶ In this role, we will revisit them in the parts on computability and complexity theory.