# Theory of Computer Science
## B10. Turing Machines I

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March 29, 2023

## B10.1 Turing Machines

## B10.2 Summary

# B10.1 Turing Machines

# Automata for Type-1 and Type-0 Languages?



Finite automata
recognize exactly the regular languages,
push-down automata exactly the
context-free languages. Are there
automata models for context-sensitive
and type-0 languages?

Yes! ⤳ Turing machines

Picture courtesy of imagerymajestic / FreeDigitalPhotos.net

# Alan Turing (1912–1954)



Picture courtesy of Jon Callas /
wikimedia commons

- ▶ British logician, mathematician, cryptanalyst and computer scientist
- ▶ most important work (for us): On Computable Numbers, with an Application to the Entscheidungsproblem ⤳ Turing machines
- ▶ collaboration on Enigma decryption
- ▶ conviction due to homosexuality; pardoned by Elizabeth II in Dec. 2013
- ▶ Turing award most important science award in computer science

# Turing Machines: Conceptually
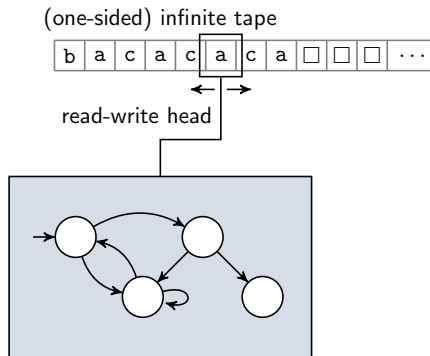
# Turing Machine: Definition

> **Definition (Deterministic Turing Machine)**
>
> A (deterministic) Turing machine (DTM) is given by a 7-tuple
> $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject} \rangle$, where
> $Q, \Sigma, \Gamma$ are all finite sets and
>
> - $Q$ is the set of states,
> - $\Sigma$ is the input alphabet, not containing the blank symbol $\square$,
> - $\Gamma$ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$,
> - $\delta : (Q \setminus \{q_{accept}, q_{reject}\}) \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function,
> - $q_0 \in Q$ is the start state,
> - $q_{accept} \in Q$ is the accept state,
> - $q_{reject} \in Q$ is the reject state, where $q_{accept} \neq q_{reject}$.

# Turing Machine: Transition Function

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ be a DTM.

> **What is the Intuitive Meaning of the Transition Function $\delta$?**
> $\delta(q, a) = \langle q', b, D \rangle$:
> - If $M$ is in state $q$ and reads $a$, then
> - $M$ transitions to state $q'$ in the next step,
> - replacing $a$ with $b$,
> - and moving the head in direction $D \in \{L, R\}$, where:
>   - R: one step to the right,
>   - L: one step to the left, except if the head is on the left-most cell of the tape in which case there is no movement
>
>
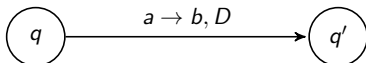
# Deterministic Turing Machine: Example

$\langle \{q_1, \ldots, q_5, q_{\text{accept}}, q_{\text{reject}}\}, \{0\}, \{0, \mathbf{x}, \square\}, \delta, q_1, q_{\text{accept}}, q_{\text{reject}} \rangle$
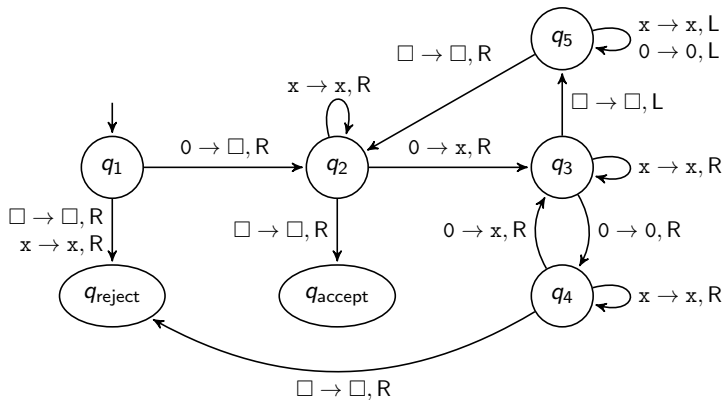
# Turing Machine: Configuration

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**Definition (Configuration of a Turing Machine)**

A configuration of a Turing machine
$M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$
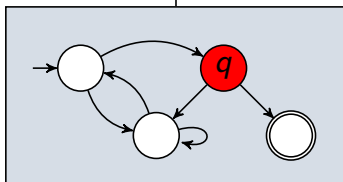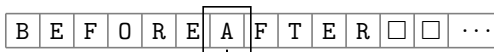is given by a triple $c \in \Gamma^* \times Q \times \Gamma^+$.

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Configuration $\langle w_1, q, w_2 \rangle$ intuitively means that

▶ the non-empty or already visited part of the tape contains the word $w_1 w_2$,

▶ the read-write head is on the first symbol of $w_2$, and

▶ the TM is in state $q$.

# Turing Machine Configurations: Example

## Example

configuration $\langle \text{BEFORE}, q, \text{AFTER}\square\square \rangle$.

# Turing Machine Configurations: Start Configuration

Initially

▶ the TM is in start state $q_0$,

▶ the head is on the first tape cell, and

▶ the tape contains the input word $w$ followed by an infinite number of $\square$ entries.

The corresponding start configuration is $\langle \varepsilon, q_0, w \rangle$ if $w \neq \varepsilon$ and $\langle \varepsilon, q_0, \square \rangle$ if $w = \varepsilon$.
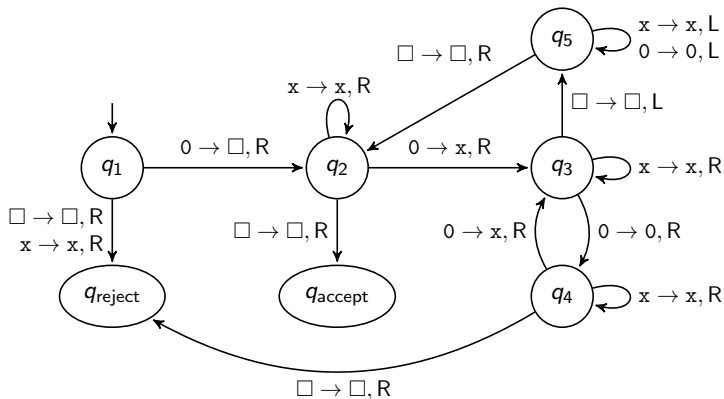
## Turing Machine: Step

> ### Definition (Transition/Step of a Turing Machine)
> A DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ transitions
> from configuration $c$ to configuration $c'$ in one step ($c \vdash_M c'$)
> according to the following rules:
>
> ▶ $\langle a_1 \ldots a_m, q, b_1 \ldots b_n \rangle \vdash_M \langle a_1 \ldots a_m c, q', b_2 \ldots b_n \rangle$
>    if $\delta(q, b_1) = \langle q', c, \mathsf{R} \rangle$, $m \geq 0$, $n \geq 2$
>
> ▶ $\langle a_1 \ldots a_m, q, b_1 \rangle \vdash_M \langle a_1 \ldots a_m c, q', \square \rangle$
>    if $\delta(q, b_1) = \langle q', c, \mathsf{R} \rangle$, $m \geq 0$
>
> ▶ $\langle a_1 \ldots a_m, q, b_1 \ldots b_n \rangle \vdash_M \langle a_1 \ldots a_{m-1}, q', a_m c b_2 \ldots b_n \rangle$
>    if $\delta(q, b_1) = \langle q', c, \mathsf{L} \rangle$, $m \geq 1$, $n \geq 1$
>
> ▶ $\langle \varepsilon, q, b_1 \ldots b_n \rangle \vdash_M \langle \varepsilon, q', c b_2 \ldots b_n \rangle$
>    if $\delta(q, b_1) = \langle q', c, \mathsf{L} \rangle$, $n \geq 1$

## Step: Exercise (Slido)



$\langle \Box \mathrm{x}, q_3, 00 \rangle \vdash \ ?$

# DTM: Accepted Words

Intuitively, a DTM accepts a word if its computation terminates in the accept state.

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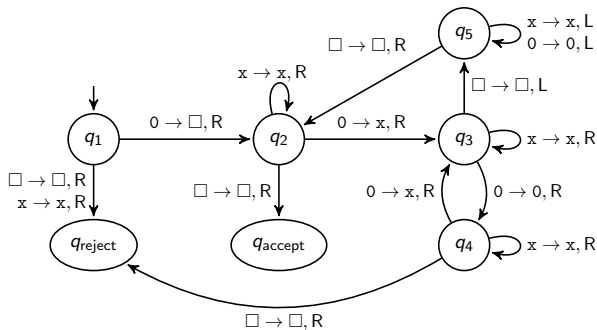**Definition (Words Accepted by a DTM)**

DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject} \rangle$ accepts the word
$w = a_1 \ldots a_n$ if there is a sequence of configurations $c_0, \ldots, c_k$ with

1. $c_0$ is the start configuration of $M$ on input $w$,

2. $c_i \vdash_M c_{i+1}$ for all $i \in \{0, \ldots, k-1\}$, and

3. $c_k$ is an accepting configuration,
   i.e., a configuration with state $q_{accept}$.

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## Accepted Word: Example (Slido)

Does this Turing machine accept input 0000?

# DTM: Recognized Language
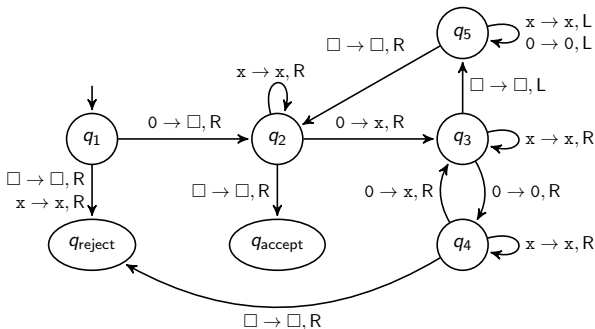
> ### Definition (Language Recognized by a DTM)
> Let $M$ be a deterministic Turing Machine
> The language recognized by $M$ (or the language of $M$) is defined
> as $\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

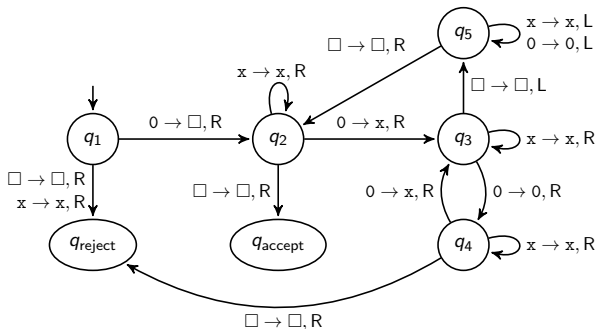> ### Definition (Turing-recognizable Language)
> We call a language Turing-recognizable if some deterministic
> Turing machine recognizes it.

## Turing Machine: Example



1. Sweep left to right across the tape, crossing off every other 0.
2. If in stage 1 the tape contained a single 0, accept.
3. If in stage 1 the tape contained more than one 0 and the number of 0s was odd, reject.
4. Return the head to the left end of the tape and go to stage 1.

# Recognized Language: Example



What language does the Turing machine recognize?

## Deciders

- A Turing machine either fails to accept an input
  - because it rejects it (entering $q_{reject}$) or
  - because it loops (= does not halt).
- A Turing machine that halts on all inputs (entering $q_{reject}$ or $q_{accept}$) is called a decider.
- A decider that recognizes some language also is said to decide the language.

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Definition (Turing-decidable Language)

We call a language Turing-decidable (or decidable) if some deterministic Turing machine decides it.

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# Exercise (if time)

Specify the state diagram of a DTM that decides language

$$L = \{w\#w \mid w \in \{0, 1\}^*\}.$$

Feel free to solve this together with your neighbour.

# B10.2 Summary

# Summary

- ▶ Turing machines only have finitely many states but an <span style="color:red">unbounded tape</span> as "memory".
- ▶ Alan Turing proposed them as a mathematical model for arbitrary algorithmic computations.
- ▶ In this role, we will revisit them in the parts on computability and complexity theory.