Theory of Computer Science

B2. Grammars

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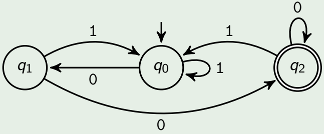
Introduction

Reminder: Alphabets and Formal Languages

- An alphabet Σ is a finite non-empty set of symbols.
- A word over Σ is a finite sequence of elements from Σ.
- The empty word is denoted by ε.
- Σ^* denotes the set of all words over Σ.
- Σ^+ denotes the set of all non-empty words over Σ.
- A formal language (over alphabet Σ) is a subset of Σ^*.

Reminder: Finite Automata and Formal Languages

Example



The DFA recognizes the language $\{w \in \{0, 1\}^* \mid w \text{ ends with } 00\}$.

- A finite automaton defines a language, the language it recognizes.
- The specification of the automaton is always finite.
- The recognized language can be infinite.

Other Ways to Specify Formal Languages?

Sought: General concepts to define
(often infinite) formal languages
with finite descriptions.

- today: grammars
- later: more automata, regular expressions, ...

Grammar: Example

Variables $V = \{\mathsf{S}, \mathsf{X}, \mathsf{Y}\}$
Alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.
Production rules:

$$\mathsf{S} \to \varepsilon \qquad \mathsf{X} \to \mathtt{a}\mathsf{X}\mathsf{Y}\mathtt{c} \qquad \mathtt{c}\mathsf{Y} \to \mathsf{Y}\mathtt{c}$$
$$\mathsf{S} \to \mathtt{abc} \qquad \mathsf{X} \to \mathtt{abc} \qquad \mathtt{b}\mathsf{Y} \to \mathtt{bb}$$
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You start from S and may in each step replace the left-hand side of a rule with the right-hand side of the same rule. This way, derive a word over Σ.

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Exercise

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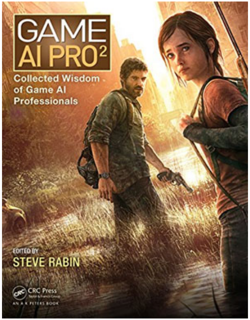
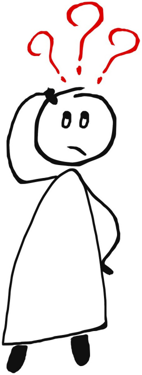
Derive word $\mathtt{aabbcc}$ starting from S.

Application: Content Generation in Games

- http://www.gameaipro.com/
- GameAIPro 2, chapter 40
  Procedural Content Generation: An Overview by Gillian Smith



Questions



Questions?

Grammars

Grammars

Definition (Grammars)

A grammar is a 4-tuple $\langle V, \Sigma, R, S\rangle$ with:

- V finite set of variables (nonterminal symbols)
- Σ finite alphabet of terminal symbols with $V \cap \Sigma = \emptyset$
- $R \subseteq (V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ finite set of rules
- $S \in V$ start variable

A rule is sometimes also called a production or a production rule.

Rule Sets

What exactly does $R \subseteq (V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ mean?

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- Instead of $\langle x, y \rangle$ we usually write rules in the form $x \rightarrow y$.

Rules: Examples

Example

Let $\Sigma = \{a, b, c\}$ and $V = \{X, Y, Z\}$.

Some examples of rules in $(V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$:

$$\begin{aligned} X &\rightarrow X\mathtt{a}Y \\ Y\mathtt{b} &\rightarrow \mathtt{a} \\ XY &\rightarrow \varepsilon \\ XYZ &\rightarrow \mathtt{abc} \\ \mathtt{abXc} &\rightarrow XYZ \end{aligned}$$

Derivations

Definition (Derivations)

Let $\langle V, \Sigma, R, S\rangle$ be a grammar. A word $v \in (V \cup \Sigma)^*$ can be derived from word $u \in (V \cup \Sigma)^+$ (written as $u \Rightarrow v$) if

1. $u = xyz$, $v = xy'z$ with $x, z \in (V \cup \Sigma)^*$ and
2. there is a rule $y \to y' \in R$.

We write: $u \Rightarrow^* v$ if v can be derived from u in finitely many steps (i. e., by using n derivations for $n \in \mathbb{N}_0$).

Language Generated by a Grammar

Definition (Languages)

The language generated by a grammar $G = \langle V, \Sigma, P, S \rangle$

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

is the set of all words from Σ^* that can be derived from S with finitely many rule applications.

Grammars

Example (Languages over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$)

- $L_1 = \{\mathtt{a}, \mathtt{aa}, \mathtt{aaa}, \mathtt{aaaa}, \dots\} = \{a\}^+$

Example grammars: blackboard

Grammars

Example (Languages over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$)

- $L_2 = \Sigma^*$

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Grammars

Example (Languages over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$)

- $L_3 = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\} = \{\varepsilon, \mathtt{ab}, \mathtt{aabb}, \mathtt{aaabbb}, \dots\}$

Example grammars: blackboard

Grammars

Example (Languages over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$)

- $L_4 = \{\varepsilon\}$

Example grammars: blackboard

Grammars

Example (Languages over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$)

- $L_5 = \emptyset$

Example grammars: blackboard

Grammars

Example (Languages over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$)

- $L_6 = \{w \in \Sigma^* \mid w \text{ contains twice as many as as bs}\}$
  $= \{\varepsilon, \mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \dots\}$

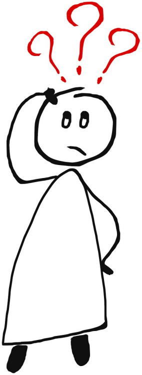
Example grammars: blackboard

Exercise

Specify a grammar that generates language

$$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid |w| = 3\}.$$

Questions



Questions?

Chomsky Hierarchy

Noam Chomsky

- Avram Noam Chomsky (*1928)
- "the father of modern linguistics"
- American linguist, philosopher, cognitive scientist, social critic, and political activist

- combined linguistics, cognitive science and computer science
- opponent of U.S. involvement in the Vietnam war
- there is a wikipedia page solemnly on his political positions

→ Organized grammars into the Chomsky hierarchy.

Chomsky Hierarchy

Definition (Chomsky Hierarchy)

- Every grammar is of type 0 (all rules allowed).
- Grammar is of type 1 (context-sensitive) if all rules are of the form $\alpha B\gamma \to \alpha\beta\gamma$ with $B \in V$ and $\alpha, \gamma \in (V \cup \Sigma)^*$ and $\beta \in (V \cup \Sigma)^+$
- Grammar is of type 2 (context-free) if all rules are of the form $A \to w$, where $A \in V$ and $w \in (V \cup \Sigma)^+$.
- Grammar is of type 3 (regular) if all rules are of the form $A \to w$, where $A \in V$ and $w \in \Sigma \cup \Sigma V$.

special case: rule $S \to \varepsilon$ is always allowed if S is the start variable and never occurs on the right-hand side of any rule.

Chomsky Hierarchy: Examples

Examples: blackboard

Chomsky Hierarchy

Definition (Type 0–3 Languages)

A language $L \subseteq \Sigma^*$ is of type 0 (type 1, type 2, type 3) if there exists a type-0 (type-1, type-2, type-3) grammar G with $\mathcal{L}(G) = L$.

Type k Language: Example (slido)

Example

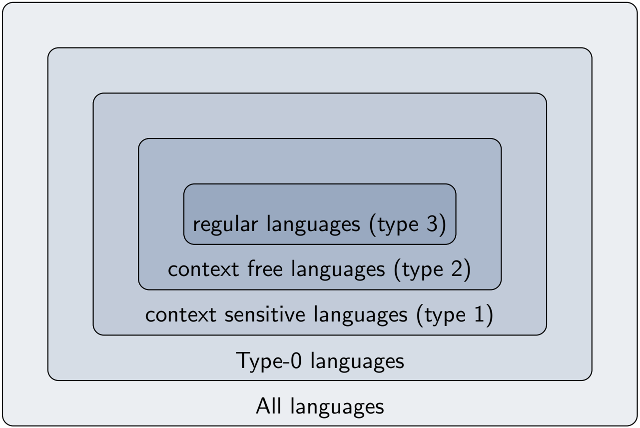
Consider the language L generated by the grammar
$\langle\{\mathrm{F}, \mathrm{A}, \mathrm{N}, \mathrm{C}, \mathrm{D}\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \neg, \wedge, \vee, (,)\}, R, \mathrm{F}\rangle$
with the following rules R:

$$
\begin{array}{lll}
\mathrm{F} \to \mathrm{A} & \mathrm{A} \to \mathtt{a} & \mathrm{N} \to \neg\mathrm{F} \\
\mathrm{F} \to \mathrm{N} & \mathrm{A} \to \mathtt{b} & \mathrm{C} \to (\mathrm{F} \wedge \mathrm{F}) \\
\mathrm{F} \to \mathrm{C} & \mathrm{A} \to \mathtt{c} & \mathrm{D} \to (\mathrm{F} \vee \mathrm{F}) \\
\mathrm{F} \to \mathrm{D} & &
\end{array}
$$

Questions:

- Is L a type-0 language?
- Is L a type-1 language?
- Is L a type-2 language?
- Is L a type-3 language?

Chomsky Hierarchy



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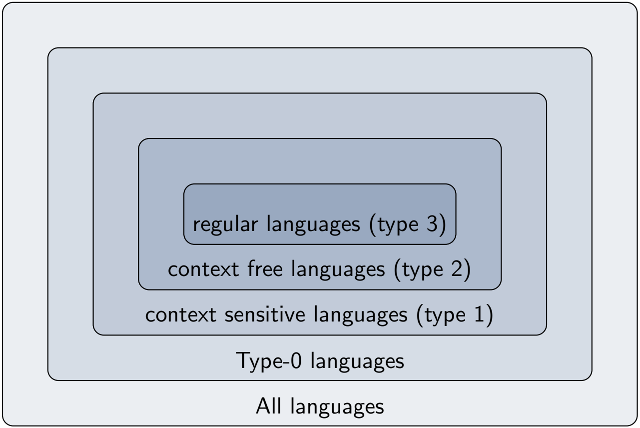

Note: Not all languages can be described by grammars. (Proof?)

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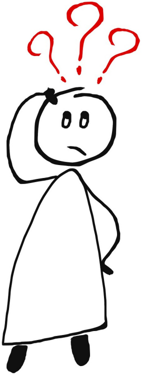

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Summary

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- Languages are sets of symbol sequences.
- Grammars are one possible way to specify languages.
- Language generated by a grammar is the set of all words (of terminal symbols) derivable from the start symbol.
- Chomsky hierarchy distinguishes between languages at different levels of expressiveness.