Theory of Computer Science

B2. Grammars

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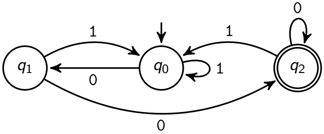
B2.1 Introduction

Reminder: Alphabets and Formal Languages

- An alphabet Σ is a finite non-empty set of symbols.
- A word over Σ is a finite sequence of elements from Σ.
- The empty word is denoted by ε.
- Σ^* denotes the set of all words over Σ.
- Σ^+ denotes the set of all non-empty words over Σ.
- A formal language (over alphabet Σ) is a subset of Σ^*.

Reminder: Finite Automata and Formal Languages

Example



The DFA recognizes the language $\{w \in \{0,1\}^* \mid w \text{ ends with } 00\}$.

- A finite automaton defines a language, the language it recognizes.
- The specification of the automaton is always finite.
- The recognized language can be infinite.

Other Ways to Specify Formal Languages?

Sought: General concepts to define
(often infinite) formal languages
with finite descriptions.

- today: grammars
- later: more automata, regular expressions, ...

Grammar: Example

Variables $V = \{\mathsf{S}, \mathsf{X}, \mathsf{Y}\}$
Alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.
Production rules:

$$\mathsf{S} \to \varepsilon \qquad \mathsf{X} \to \mathtt{a}\mathsf{X}\mathsf{Y}\mathtt{c} \qquad \mathtt{c}\mathsf{Y} \to \mathsf{Y}\mathtt{c}$$

$$\mathsf{S} \to \mathtt{abc} \qquad \mathsf{X} \to \mathtt{abc} \qquad \mathtt{b}\mathsf{Y} \to \mathtt{bb}$$

$$\mathsf{S} \to \mathsf{X}$$

You start from S and may in each step replace the left-hand side of a rule with the right-hand side of the same rule. This way, derive a word over Σ.

Exercise

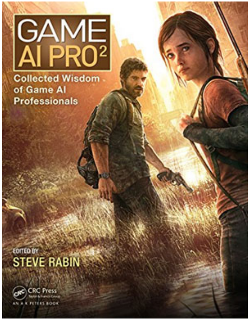
Variables $V = \{S, X, Y\}$
Alphabet $\Sigma = \{a, b, c\}$.
Production rules:

$S \rightarrow \varepsilon$ $X \rightarrow aXYc$ $cY \rightarrow Yc$
$S \rightarrow abc$ $X \rightarrow abc$ $bY \rightarrow bb$
$S \rightarrow X$

Derive word aabbcc starting from S.

Application: Content Generation in Games

- http://www.gameaipro.com/
- GameAIPro 2, chapter 40
  Procedural Content Generation:
  An Overview by Gillian Smith



B2.2 Grammars

Grammars

Definition (Grammars)

A grammar is a 4-tuple $\langle V, \Sigma, R, S \rangle$ with:

- V finite set of variables (nonterminal symbols)
- Σ finite alphabet of terminal symbols with $V \cap \Sigma = \emptyset$
- $R \subseteq (V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ finite set of rules
- $S \in V$ start variable

A rule is sometimes also called a production or a production rule.

Rule Sets

What exactly does $R \subseteq (V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ mean?

- $(V \cup \Sigma)^*$: all words over $(V \cup \Sigma)$
- for languages L and L', their concatenation is the language $LL' = \{xy \mid x \in L \text{ and } y \in L'\}$.
- $(V \cup \Sigma)^* V (V \cup \Sigma)^*$: words composed from
  - a word over $(V \cup \Sigma)$,
  - followed by a single variable symbol,
  - followed by a word over $(V \cup \Sigma)$

  $\rightarrow$ word over $(V \cup \Sigma)$ containing at least one variable symbol
- $\times$: Cartesian product
- $(V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$: set of all pairs $\langle x, y \rangle$, where x word over $(V \cup \Sigma)$ with at least one variable and y word over $(V \cup \Sigma)$
- Instead of $\langle x, y \rangle$ we usually write rules in the form $x \rightarrow y$.

Rules: Examples

Example

Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $V = \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$.

Some examples of rules in $(V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$:

$$\begin{aligned}
\mathsf{X} &\to \mathsf{X}\mathtt{a}\mathsf{Y} \\
\mathsf{Y}\mathtt{b} &\to \mathtt{a} \\
\mathsf{XY} &\to \varepsilon \\
\mathsf{XYZ} &\to \mathtt{abc} \\
\mathtt{abXc} &\to \mathsf{XYZ}
\end{aligned}$$

Derivations

Definition (Derivations)

Let $\langle V, \Sigma, R, S \rangle$ be a grammar. A word $v \in (V \cup \Sigma)^*$ can be derived from word $u \in (V \cup \Sigma)^+$ (written as $u \Rightarrow v$) if

1. $u = xyz$, $v = xy'z$ with $x, z \in (V \cup \Sigma)^*$ and
2. there is a rule $y \to y' \in R$.

We write: $u \Rightarrow^* v$ if v can be derived from u in finitely many steps (i. e., by using n derivations for $n \in \mathbb{N}_0$).

Language Generated by a Grammar

Definition (Languages)

The language generated by a grammar $G = \langle V, \Sigma, P, S \rangle$

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

is the set of all words from Σ^* that can be derived from S with finitely many rule applications.

Grammars

Example (Languages over $\Sigma = \{a, b\}$)

- $L_1 = \{a, aa, aaa, aaaa, \dots\} = \{a\}^+$
- $L_2 = \Sigma^*$
- $L_3 = \{a^n b^n \mid n \geq 0\} = \{\varepsilon, ab, aabb, aaabbb, \dots\}$
- $L_4 = \{\varepsilon\}$
- $L_5 = \emptyset$
- $L_6 = \{w \in \Sigma^* \mid w \text{ contains twice as many as as bs}\}$
  $= \{\varepsilon, aab, aba, baa, \dots\}$

Example grammars: blackboard

Exercise

Specify a grammar that generates language

$$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid |w| = 3\}.$$

B2.3 Chomsky Hierarchy

Noam Chomsky

- Avram Noam Chomsky (*1928)
- "the father of modern linguistics"
- American linguist, philosopher, cognitive scientist, social critic, and political activist

- combined linguistics, cognitive science and computer science
- opponent of U.S. involvement in the Vietnam war
- there is a wikipedia page solemnly on his political positions

→ Organized grammars into the Chomsky hierarchy.

Chomsky Hierarchy

Definition (Chomsky Hierarchy)

- Every grammar is of type 0 (all rules allowed).
- Grammar is of type 1 (context-sensitive) if all rules are of the form $\alpha B \gamma \to \alpha \beta \gamma$ with $B \in V$ and $\alpha, \gamma \in (V \cup \Sigma)^*$ and $\beta \in (V \cup \Sigma)^+$
- Grammar is of type 2 (context-free) if all rules are of the form $A \to w$, where $A \in V$ and $w \in (V \cup \Sigma)^+$.
- Grammar is of type 3 (regular) if all rules are of the form $A \to w$, where $A \in V$ and $w \in \Sigma \cup \Sigma V$.

special case: rule $S \to \varepsilon$ is always allowed if S is the start variable and never occurs on the right-hand side of any rule.

Chomsky Hierarchy

Definition (Type 0–3 Languages)

A language $L \subseteq \Sigma^*$ is of type 0 (type 1, type 2, type 3) if there exists a type-0 (type-1, type-2, type-3) grammar G with $\mathcal{L}(G) = L$.

Type k Language: Example (slido)

Example

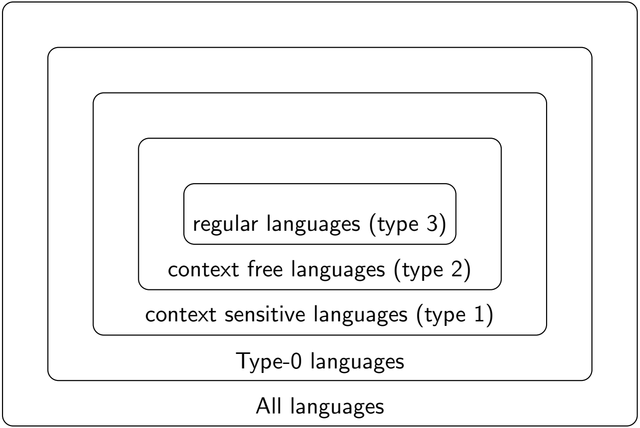
Consider the language L generated by the grammar
$\langle\{\mathsf{F}, \mathsf{A}, \mathsf{N}, \mathsf{C}, \mathsf{D}\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \neg, \wedge, \vee, (,)\}, R, \mathsf{F}\rangle$
with the following rules R:

$$\begin{array}{lll} \mathsf{F} \to \mathsf{A} & \mathsf{A} \to \mathtt{a} & \mathsf{N} \to \neg\mathsf{F} \\ \mathsf{F} \to \mathsf{N} & \mathsf{A} \to \mathtt{b} & \mathsf{C} \to (\mathsf{F} \wedge \mathsf{F}) \\ \mathsf{F} \to \mathsf{C} & \mathsf{A} \to \mathtt{c} & \mathsf{D} \to (\mathsf{F} \vee \mathsf{F}) \\ \mathsf{F} \to \mathsf{D} & & \end{array}$$

Questions:

- Is L a type-0 language?
- Is L a type-1 language?
- Is L a type-2 language?
- Is L a type-3 language?

Chomsky Hierarchy



Note: Not all languages can be described by grammars. (Proof?)

B2.4 Summary

Summary

- Languages are sets of symbol sequences.
- Grammars are one possible way to specify languages.
- Language generated by a grammar is the set of all words (of terminal symbols) derivable from the start symbol.
- Chomsky hierarchy distinguishes between languages at different levels of expressiveness.