Theory of Computer Science

B1. Finite Automata

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B1.1 Introduction

Course Contents

Parts of the course:

A. background
  ▷ mathematical foundations and proof techniques

B. automata theory and formal languages
  (Automatentheorie und formale Sprachen)
  ▷ What is a computation?

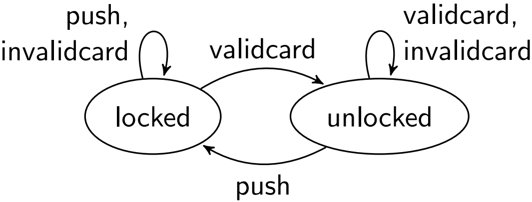
C. Turing computability (Turing-Berechenbarkeit)
  ▷ What can be computed at all?

D. complexity theory (Komplexitätstheorie)
  ▷ What can be computed efficiently?

E. more computability theory (mehr Berechenbarkeitheorie)
  ▷ Other models of computability

A Controller for a Turnstile

- simple access control
- card reader and push sensor
- card can either be valid or invalid



- Finite automata are a good model for computers with very limited memory.
  Where can the turnstile controller store information about what it has seen in the past?
- We will not consider automata that run forever but that process a finite input sequence and then classify it as accepted or not.
- Before we get into the details, we need some background on formal languages to formalize what is a valid input sequence.

B1.2 Alphabets and Formal Languages

Alphabets and Formal Languages

Definition (Alphabets, Words and Formal Languages)

An alphabet Σ is a finite non-empty set of symbols.

A word over Σ is a finite sequence of elements from Σ.
The empty word (the empty sequence of elements) is denoted by ε.
Σ^* denotes the set of all words over Σ.
Σ^+ $(= \Sigma^* \setminus \{\varepsilon\})$ denotes the set of all non-empty words over Σ.

We write $|w|$ for the length of a word w.

A formal language (over alphabet Σ) is a subset of Σ^*.

Example

$\Sigma = \{\mathtt{a}, \mathtt{b}\}$
$\Sigma^* = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{bb}, \dots\}$
$|\mathtt{aba}| = 3, |\mathtt{b}| = 1, |\varepsilon| = 0$

Languages: Examples

Example (Languages over $\Sigma = \{\mathrm{a}, \mathrm{b}\}$)

- $S_1 = \{\mathrm{a}, \mathrm{aa}, \mathrm{aaa}, \mathrm{aaaa}, \dots\} = \{a\}^+$
- $S_2 = \Sigma^*$
- $S_3 = \{\mathrm{a}^n\mathrm{b}^n \mid n \geq 0\} = \{\varepsilon, \mathrm{ab}, \mathrm{aabb}, \mathrm{aaabbb}, \dots\}$
- $S_4 = \{\varepsilon\}$
- $S_5 = \emptyset$
- $S_6 = \{w \in \Sigma^* \mid w \text{ contains twice as many as as bs}\}$
  $= \{\varepsilon, \mathrm{aab}, \mathrm{aba}, \mathrm{baa}, \dots\}$
- $S_7 = \{w \in \Sigma^* \mid |w| = 3\}$
  $= \{\mathrm{aaa}, \mathrm{aab}, \mathrm{aba}, \mathrm{baa}, \mathrm{bba}, \mathrm{bab}, \mathrm{abb}, \mathrm{bbb}\}$

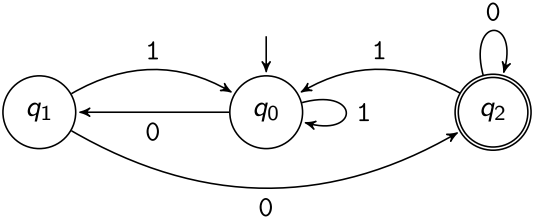
Exercise (slido)

Consider $\Sigma = \{\text{push}, \text{validcard}\}$.

What is $|\text{pushvalidcard}|$?

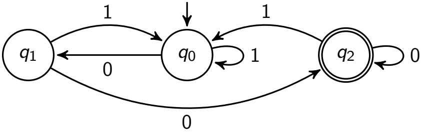
B1.3 DFAs

Finite Automaton: Example



When reading the input 01100 the automaton visits the states q_0, q_1, q_0, q_0, q_1, q_2.

Finite Automata: Terminology and Notation



- states $Q = \{q_0, q_1, q_2\}$
- input alphabet $\Sigma = \{0, 1\}$
- transition function δ
- start state q_0
- accept states $\{q_2\}$

$\delta(q_0, 0) = q_1$

$\delta(q_0, 1) = q_0$

$\delta(q_1, 0) = q_2$

$\delta(q_1, 1) = q_0$

$\delta(q_2, 0) = q_2$

$\delta(q_2, 1) = q_0$

δ	0	1
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_2	q_0

table form of δ

Deterministic Finite Automaton: Definition

Definition (Deterministic Finite Automata)

A deterministic finite automaton (DFA) is a 5-tuple $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ where

- Q is the finite set of states
- Σ is the input alphabet
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states (or final states)

DFA: Accepted Words

Intuitively, a DFA accepts a word if its computation terminates in an accept state.

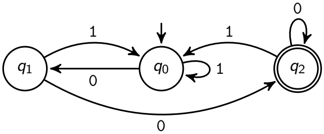
Definition (Words Accepted by a DFA)

DFA $M = \langle Q, \Sigma, \delta, q_0, F\rangle$ accepts the word $w = a_1 \dots a_n$ if there is a sequence of states $q'_0, \dots, q'_n \in Q$ with

1. $q'_0 = q_0$,
2. $\delta(q'_{i-1}, a_i) = q'_i$ for all $i \in \{1, \dots, n\}$ and
3. $q'_n \in F$.

Example

Example

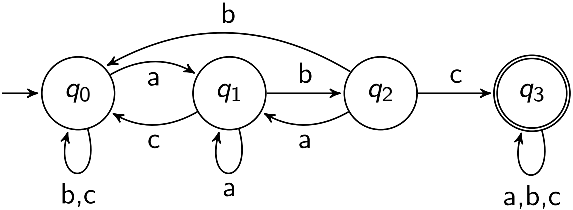


accepts:
00
10010100
01000

does not accept:
ε
1001010
010001

Exercise (slido)

Consider the following DFA:



Which of the following words does it accept?

- abc
- ababcb
- babbc

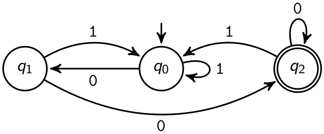
DFA: Recognized Language

Definition (Language Recognized by a DFA)

Let M be a deterministic finite automaton.
The language recognized by M is defined as
$\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

Example

Example



The DFA recognizes the language $\{w \in \{0,1\}^* \mid w \text{ ends with } 00\}$.

A Note on Terminology

- In the literature, "accept" and "recognize" are sometimes used synonymously or the other way around.
  DFA recognizes a word or accepts a language.
- We try to stay consistent using the previous definitions (following the text book by Sipser).

B1.4 NFAs

Nondeterministic Finite Automata

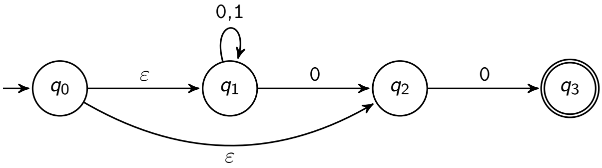
Why are DFAs called deterministic automata? What are nondeterministic automata, then?

Picture courtesy of stockimages / FreeDigitalPhotos.net

In what Sense is a DFA Deterministic?

- A DFA has a single fixed state from which the computation starts.
- When a DFA is in a specific state and reads an input symbol, we know what the next state will be.
- For a given input, the entire computation is determined.
- This is a deterministic computation.

Nondeterministic Finite Automata: Example



differences to DFAs:

- transition function δ can lead to zero or more successor states for the same $a \in \Sigma$
- ε-transitions can be taken without "consuming" a symbol from the input
- the automaton accepts a word if there is at least one accepting sequence of states

Nondeterministic Finite Automaton: Definition

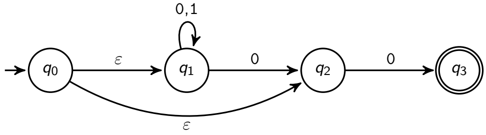
Definition (Nondeterministic Finite Automata)

A nondeterministic finite automaton (NFA) is a 5-tuple $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ where

- Q is the finite set of states
- Σ is the input alphabet
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Q)$ is the transition function (mapping to the power set of Q)
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

DFAs are (essentially) a special case of NFAs.

Accepting Computation: Example



$w = 0100$

⇝ computation tree on blackboard

ε-closure of a State

For a state $q \in Q$, we write $E(q)$ to denote the set of states that are reachable from q via ε-transitions in δ.

Definition (ε-closure)

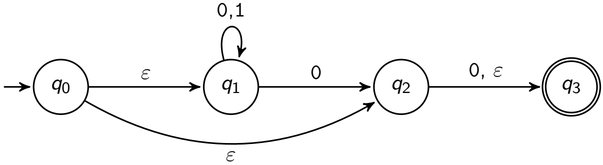
For NFA $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ and state $q \in Q$, state p is in the ε-closure $E(q)$ of q iff there is a sequence of states $q'_0, \dots, q'_n$ with

1. $q'_0 = q$,
2. $q'_i \in \delta(q'_{i-1}, \varepsilon)$ for all $i \in \{1, \dots, n\}$ and
3. $q'_n = p$.

$q \in E(q)$ for every state q

Exercise (slido)

Consider the following NFA:



Which states are in the ε-closure $E(q_0)$?

- q_0
- q_1
- q_2
- q_3

NFA: Accepted Words

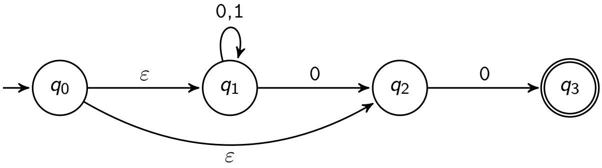
Definition (Words Accepted by an NFA)

NFA $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ **accepts the word** $w = a_1 \dots a_n$ if there is a sequence of states $q'_0, \dots, q'_n \in Q$ with

1. $q'_0 \in E(q_0)$,
2. $q'_i \in \bigcup_{q \in \delta(q'_{i-1}, a_i)} E(q)$ for all $i \in \{1, \dots, n\}$ and
3. $q'_n \in F$.

Example: Accepted Words

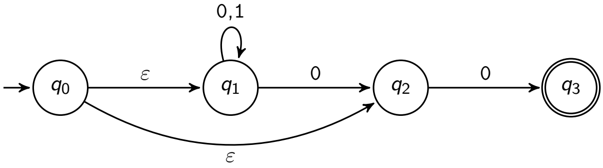
Example



accepts:
0
10010100
01000

does not accept:
ε
1001010
010001

Exercise (slido)



Does this NFA accept input 01010?

NFA: Recognized Language

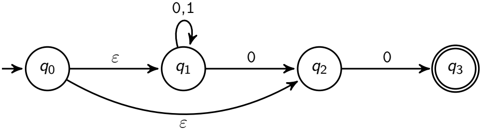
Definition (Language Recognized by an NFA)

Let M be an NFA with input alphabet Σ.

The language recognized by M is defined as
$\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

Example: Recognized Language

Example



The NFA recognizes the language
$\{w \in \{0, 1\}^* \mid w = 0$ or w ends with $00\}$.

B1.5 DFAs vs. NFAs

DFAs are No More Powerful than NFAs

Observation

Every language recognized by a DFA is also recognized by an NFA.

We can transform a DFA into an NFA by replacing every transition $\delta(q, a) = q'$ with $\delta(q, a) = \{q'\}$.

Question

DFAs are
no more powerful than NFAs.
But are there languages
that can be recognized
by an NFA but not by a DFA?

Picture courtesy of imagerymajestic / FreeDigitalPhotos.net

NFAs are No More Powerful than DFAs

Theorem (Rabin, Scott)

Every language recognized by an NFA is also recognized by a DFA.

The proof of the theorem is constructive and shows how we can convert an NFA to an equivalent DFA. Let's first have a look at the idea by means of an example (on the blackboard).

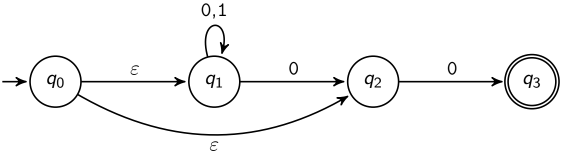
NFAs are No More Powerful than DFAs

Theorem (Rabin, Scott)

Every language recognized by an NFA is also recognized by a DFA.

The proof of the theorem is constructive and shows how we can convert an NFA to an equivalent DFA. Let's first have a look at the idea by means of an example (on the blackboard).

Conversion of an NFA to an Equivalent DFA: Example



NFAs are No More Powerful than DFAs

Theorem (Rabin, Scott)

Every language recognized by an NFA is also recognized by a DFA.

Proof.

For every NFA $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ we can construct a DFA $M' = \langle Q', \Sigma, \delta', q'_0, F' \rangle$ with $\mathcal{L}(M) = \mathcal{L}(M')$.
Here M' is defined as follows:

- $Q' := \mathcal{P}(Q)$ (the power set of Q)
- $q'_0 := E(q_0)$
- $F' := \{\mathcal{Q} \subseteq Q \mid \mathcal{Q} \cap F \neq \emptyset\}$
- For all $\mathcal{Q} \in Q'$: $\delta'(\mathcal{Q}, a) := \bigcup_{q \in \mathcal{Q}} \bigcup_{q' \in \delta(q,a)} E(q')$

...

NFAs are No More Powerful than DFAs

Theorem (Rabin, Scott)

Every language recognized by an NFA is also recognized by a DFA.

Proof (continued).

For every $w = a_1 a_2 \dots a_n \in \Sigma^*$:

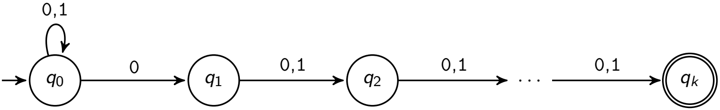
$w \in \mathcal{L}(M)$
iff there is a sequence of states $p_0, p_1, \dots, p_n$ with
$p_0 \in E(q_0)$, $p_n \in F$ and
$p_i \in \bigcup_{q \in \delta(p_{i-1}, a_i)} E(q)$ for all $i \in \{1, \dots, n\}$
iff there is a sequence of subsets $\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_n$ with
$\mathcal{Q}_0 = q_0'$, $\mathcal{Q}_n \in F'$ and $\delta'(\mathcal{Q}_{i-1}, a_i) = \mathcal{Q}_i$ for all $i \in \{1, \dots, n\}$
iff $w \in \mathcal{L}(M')$ □

NFAs are More Compact than DFAs

Example

For $k \geq 1$ consider the language
$L_k = \{w \in \{0, 1\}^* \mid |w| \geq k$ and the k-th last symbol of w is $0\}$.

The language L_k can be accepted by an NFA with $k + 1$ states:



There is no DFA with less than 2^k states that accepts L_k (without proof).

NFAs can often represent languages more compactly than DFAs.

B1.6 Summary

Summary

- DFAs are automata where every state transition is uniquely determined.
- NFAs can have zero, one or more transitions for a given state and input symbol.
- NFAs can have ϵ-transitions that can be taken without reading a symbol from the input.
- NFAs accept a word if there is at least one accepting sequence of states.
- DFAs and NFAs accept the same languages.