Theory of Computer Science

A3. Proof Techniques

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Introduction

What is a Proof?

A mathematical proof is

- a sequence of logical steps
- starting with one set of statements
- that comes to the conlusion
  that some statement must be true.

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What is a statement?

Mathematical Statements

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The statement is true if the conclusions are true whenever the preconditions are true.

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Notes:

- set of preconditions is sometimes empty
- often, “assumptions” is used instead of “preconditions”; slightly unfortunate because “assumption” is also used with another meaning ($\rightsquigarrow$ cf. indirect proofs)

Examples of Mathematical Statements

Examples (some true, some false):

- "Let $p \in \mathbb{N}_0$ be a prime number. Then p is odd."
- "There exists an even prime number."
- "Let $p \in \mathbb{N}_0$ with $p \geq 3$ be a prime number. Then p is odd."
- "All prime numbers $p \geq 3$ are odd."
- "For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$"

What are the preconditions, what are the conclusions?

On what Statements can we Build the Proof?

A mathematical proof is

- a sequence of logical steps
- starting with one set of statements
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  that some statement must be true.

We can use:

- axioms: statements that are assumed to always be true
  in the current context
- theorems and lemmas: statements that were already proven
  - lemma: an intermediate tool
  - theorem: itself a relevant result
- premises: assumptions we make to see
  what consequences they have

What is a Logical Step?

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Each step directly follows

- from the axioms,
- premises,
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- the preconditions of the statement we want to prove.

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For a formal definition, we would need formal logics.

The Role of Definitions

Definition

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- A definition introduces an abbreviation.
- Whenever we say "set", we could instead say "an unordered collection of distinct objects" and vice versa.
- Definitions can also introduce notation.

Disproofs

- A disproof (refutation) shows that a given mathematical statement is false by giving an example where the preconditions are true, but the conclusion is false.
- This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

- Formally, disproofs are proofs of modified ("negated") statements.
- Be careful about how to negate a statement!

Proof Strategies

typical proof/disproof strategies:

1. "All $x \in S$ with the property P also have the property Q."
   "For all $x \in S$: if x has property P, then x has property Q."
   - To prove, assume you are given an arbitrary $x \in S$ that has the property P.
     Give a sequence of proof steps showing that x must have the property Q.
   - To disprove, find a counterexample, i. e., find an $x \in S$ that has property P but not Q and prove this.

Proof Strategies

typical proof/disproof strategies:

2. “A is a subset of B.”
   - To prove, assume you have an arbitrary element $x \in A$ and prove that $x \in B$.
   - To disprove, find an element in $x \in A \setminus B$ and prove that $x \in A \setminus B$.

Proof Strategies

typical proof/disproof strategies:

3. "For all $x \in S$: x has property P iff x has property Q."
   ("iff": "if and only if")
   - To prove, separately prove "if P then Q" and "if Q then P".
   - To disprove, disprove "if P then Q" or disprove "if Q then P".

Proof Strategies

typical proof/disproof strategies:

4. "$A = B$", where A and B are sets.
   - To prove, separately prove "$A \subseteq B$" and "$B \subseteq A$".
   - To disprove, disprove "$A \subseteq B$" or disprove "$B \subseteq A$".

Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- proof by contrapositive
- mathematical induction

Exercise

You want to disprove the following statement with a counterexample:

If the sun is shining then all kids eat ice cream.

What properties must your counterexample have?

Direct Proof

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Direct Proof

Direct derivation of the statement by deducing or rewriting.

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

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We first show that $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$ ($\subseteq$ part):

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Let $x \in A \cap (B \cup C)$. Then by the definition of $\cap$ it holds that $x \in A$ and $x \in B \cup C$.

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Let $x \in A \cap (B \cup C)$. Then by the definition of $\cap$ it holds that $x \in A$ and $x \in B \cup C$.

We make a case distinction between $x \in B$ and $x \notin B$:

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

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If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

Otherwise, because $x \in B \cup C$ we know that $x \in C$ and thus with $x \in A$, that $x \in A \cap C$.

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Let $x \in A \cap (B \cup C)$. Then by the definition of $\cap$ it holds that $x \in A$ and $x \in B \cup C$.

We make a case distinction between $x \in B$ and $x \notin B$:

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

Otherwise, because $x \in B \cup C$ we know that $x \in C$ and thus with $x \in A$, that $x \in A \cap C$.

In both cases $x \in A \cap B$ or $x \in A \cap C$,
and we conclude $x \in (A \cap B) \cup (A \cap C)$. . . .

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

$\supseteq$ part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

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For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

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Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.
The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

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Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.
The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

If $x \notin A \cap B$ we know $x \in A \cap C$ due to $x \in (A \cap B) \cup (A \cap C)$.
This (analogously) implies $x \in A$ and $x \in C$, and hence $x \in B \cup C$ and thus $x \in A \cap (B \cup C)$.

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Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.
The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

If $x \notin A \cap B$ we know $x \in A \cap C$ due to $x \in (A \cap B) \cup (A \cap C)$.
This (analogously) implies $x \in A$ and $x \in C$, and hence $x \in B \cup C$ and thus $x \in A \cap (B \cup C)$.

In both cases we conclude $x \in A \cap (B \cup C)$. . . .

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

We have shown that every element of $A \cap (B \cup C)$
is an element of $(A \cap B) \cup (A \cap C)$ and vice versa.
Thus, both sets are equal. □

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

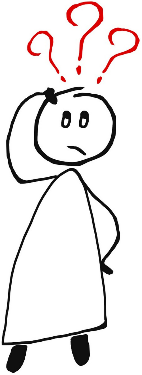
Proof.

Alternative:

$$\begin{aligned} A \cap (B \cup C) &= \{x \mid x \in A \text{ and } x \in B \cup C\} \\ &= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\} \\ &= \{x \mid (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\} \\ &= \{x \mid x \in A \cap B \text{ or } x \in A \cap C\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$

□

Questions



Questions?

Indirect Proof

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Indirect Proof (Proof by Contradiction)

- Make an assumption that the statement is false.
- Derive a contradiction from the assumption together with the preconditions of the statement.
- This shows that the assumption must be false given the preconditions of the statement, and hence the original statement must be true.

Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

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Proof.

Assumption: There are only finitely many prime numbers.

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Let $P = \{p_1, \ldots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \ldots \cdot p_n + 1$.

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Indirect Proof: Example

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Assumption: There are only finitely many prime numbers.

Let $P = \{p_1, \dots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \ldots \cdot p_n + 1$.

Since $m \geq 2$, it must have a prime factor.

Let p be such a prime factor.

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Since p is a prime number, p has to be in P.

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Define $m = p_1 \cdot \ldots \cdot p_n + 1$.

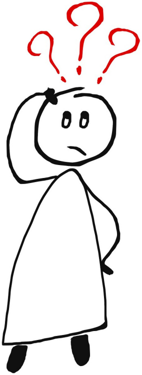
Since $m \geq 2$, it must have a prime factor.
Let p be such a prime factor.

Since p is a prime number, p has to be in P.

The number m is not divisible without remainder
by any of the numbers in P. Hence p is no factor of m.

$\rightsquigarrow$ Contradiction $\square$

Questions



Questions?

Contrapositive

Proof by Contrapositive

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Prove “If A, then B” by proving “If not B, then not A.”

Proof by Contrapositive

Proof by Contrapositive

Prove "If A, then B" by proving "If not B, then not A."

Examples:

- Prove "For all $n \in \mathbb{N}_0$: if n^2 is odd, then n is odd" by proving "For all $n \in \mathbb{N}_0$, if n is even, then n^2 is even."
- Prove "For all $n \in \mathbb{N}_0$: if n is not a square number, then $\sqrt{n}$ is irrational" by proving "For all $n \in \mathbb{N}_0$: if $\sqrt{n}$ is rational, then n is a square number."

Exercise

How would you prove the following statement by contrapositive:

If the sun is shining then all kids eat ice cream.

Mathematical Induction

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Proof of a statement for all natural numbers n with $n \geq m$

- basis: proof of the statement for $n = m$
- induction hypothesis (IH):
  suppose that the statement is true for all k with $m \leq k \leq n$
- inductive step: proof of the statement for $n + 1$
  using the induction hypothesis

Mathematical Induction: Example

Theorem

For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^{n}(2k-1) = n^2$

Mathematical Induction: Example

Theorem

For all $n \in \mathbb{N}_0$ *with* $n \geq 1$*:* $\sum_{k=1}^{n}(2k-1) = n^2$

Proof.

Mathematical induction over n:

basis $n = 1$: $\sum_{k=1}^{1}(2k-1) = 2-1 = 1 = 1^2$

□

Mathematical Induction: Example

Theorem

For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^{n}(2k-1) = n^2$

Proof.

Mathematical induction over n:

basis $n = 1$: $\sum_{k=1}^{1}(2k-1) = 2-1 = 1 = 1^2$

IH: $\sum_{k=1}^{m}(2k-1) = m^2$ for all $1 \leq m \leq n$

□

Mathematical Induction: Example

Theorem

For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^{n}(2k-1) = n^2$

Proof.

Mathematical induction over n:

basis $n = 1$: $\sum_{k=1}^{1}(2k-1) = 2 - 1 = 1 = 1^2$

IH: $\sum_{k=1}^{m}(2k-1) = m^2$ for all $1 \leq m \leq n$

inductive step $n \to n+1$:

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \Big(\sum_{k=1}^{n}(2k-1)\Big) + 2(n+1) - 1 \\ &\overset{\text{IH}}{=} n^2 + 2(n+1) - 1 \\ &= n^2 + 2n + 1 = (n+1)^2\end{aligned}$$

□

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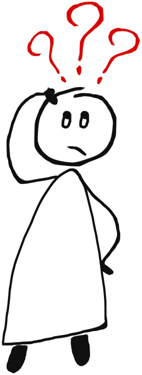

Questions?

Summary

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- A proof is based on axioms and previously proven statements.
- Individual proof steps must be obvious derivations.
- direct proof: sequence of derivations or rewriting
- indirect proof: refute the negated statement
- contrapositive: prove "$A \Rightarrow B$" as "not $B \Rightarrow$ not A"
- mathematical induction: prove statement for a starting point and show that it always carries over to the next number