

# Theory of Computer Science

## A2. Mathematical Foundations

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# Sets, Tuples, Relations

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- **cardinality**  $|M|$  of a finite set  $M$ : number of elements in  $M$

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- $A \subseteq B$ :  $A$  is a **subset** of  $B$ ,  
i. e., every element of  $A$  is an element of  $B$
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- Cardinality of power set of finite set  $S$ :  $|\mathcal{P}(S)| =$

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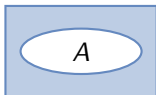
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- **complement**  $\bar{A} = B \setminus A$ , where  $A \subseteq B$  and  $B$  is the set of all considered objects (in a given context)



# Tuples

- **$k$ -tuple**: ordered sequence of  $k$  objects
- written  $(o_1, \dots, o_k)$  or  $\langle o_1, \dots, o_k \rangle$
- unlike sets, **order matters** ( $\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$ )
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- objects may occur multiple times in a tuple
- objects contained in tuples are called **components**
- terminology:
  - $k = 2$ : (ordered) pair
  - $k = 3$ : triple
- if  $k$  is clear from context (or does not matter), often just called **tuple**



# Cartesian Product

- for sets  $M_1, M_2, \dots, M_n$ , the **Cartesian product**  $M_1 \times \dots \times M_n$  is the set  
 $M_1 \times \dots \times M_n = \{\langle o_1, \dots, o_n \rangle \mid o_1 \in M_1, \dots, o_n \in M_n\}$ .
- Example:  $M_1 = \{a, b, c\}, M_2 = \{1, 2\}$ ,  
 $M_1 \times M_2 = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle\}$

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 $M_1 \times M_2 = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle\}$
- special case:  $M^k = M \times \dots \times M$  ( $k$  times)
- example with  $M = \{1, 2\}$ :  
 $M^2 = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}$

# Relations

- an  $n$ -ary **relation**  $R$  over the sets  $M_1, \dots, M_n$  is a subset of their Cartesian product:  $R \subseteq M_1 \times \dots \times M_n$ .
- example with  $M = \{1, 2\}$ :  
 $R_{\leq} \subseteq M^2$  as  $R_{\leq} = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 2 \rangle\}$

# Functions

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## Definition (Total Function)

A (total) **function**  $f : D \rightarrow C$  (with sets  $D, C$ ) maps **every value** of its **domain**  $D$  to **exactly one value** of its **codomain**  $C$ .

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- $add : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  with  $add(x, y) = x + y$
- $add_{\mathbb{R}} : \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $add_{\mathbb{R}}(x, y) = x + y$



# Functions: Example

## Example

Let  $Q = \{q_0, q_1, q_2, q_{\text{accept}}, q_{\text{reject}}\}$  and  $\Gamma = \{0, 1, \square\}$ .

Define  $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$  by

$\delta$	0	1	$\square$
$q_0$	$\langle q_0, 0, R \rangle$	$\langle q_0, 1, R \rangle$	$\langle q_1, \square, L \rangle$
$q_1$	$\langle q_2, 1, L \rangle$	$\langle q_1, 0, L \rangle$	$\langle q_{\text{reject}}, 1, L \rangle$
$q_2$	$\langle q_2, 0, L \rangle$	$\langle q_2, 1, L \rangle$	$\langle q_{\text{accept}}, \square, R \rangle$

Then, e. g.,  $\delta(q_0, 1) = \langle q_0, 1, R \rangle$

# Partial Functions

## Definition (Partial Function)

A **partial function**  $f : X \rightarrow_p Y$  maps every value in  $X$  to **at most** one value in  $Y$ .

If  $f$  does not map  $x \in X$  to any value in  $Y$ , then  $f$  is **undefined** for  $x$ .

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## Example

$f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow_p \mathbb{N}_0$  with

$$f(x, y) = \begin{cases} x - y & \text{if } y \leq x \\ \text{undefined} & \text{otherwise} \end{cases}$$

# Summary

# Summary

- **sets:** unordered, contain every element at most once
- **tuples:** ordered, can contain the same object multiple times
- **Cartesian product:**  $M_1 \times \cdots \times M_n$  set of all  $n$ -tuples where the  $i$ -th component is in  $M_i$
- **function**  $f : X \rightarrow Y$  maps every value in  $X$  to exactly one value in  $Y$
- **partial function**  $g : X \rightarrow_p Y$  may be undefined for some values in  $X$