

Theory of Computer Science

A2. Mathematical Foundations

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A2.1 Sets, Tuples, Relations

A2.2 Functions

A2.3 Summary

A2.1 Sets, Tuples, Relations

Sets

- ▶ **set**: **unordered collection** of distinguishable objects; each object contained **at most once**
- ▶ notations:
 - ▶ **explicit**, listing all elements, e. g. $A = \{1, 2, 3\}$
 - ▶ **implicit**, specifying a **property** characterizing all elements, e. g. $A = \{x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 3\}$
 - ▶ **implicit**, as a **sequence with dots**, e. g. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- ▶ $e \in M$: e is in set M (an **element** of the set)
- ▶ $e \notin M$: e is not in set M
- ▶ **empty set** $\emptyset = \{\}$
- ▶ **cardinality** $|M|$ of a finite set M : number of elements in M

Sets

- ▶ $A \subseteq B$: A is a **subset** of B ,
i. e., every element of A is an element of B
- ▶ $A \subset B$: A is a **strict subset** of B ,
i. e., $A \subseteq B$ and $A \neq B$.
- ▶ **power set** $\mathcal{P}(M)$: set of all subsets of M
e. g., $\mathcal{P}(\{a, b\}) =$
- ▶ Cardinality of power set of finite set S : $|\mathcal{P}(S)| =$

Set Operations

- ▶ **intersection** $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



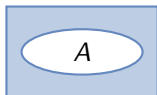
- ▶ **union** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



- ▶ **difference** $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



- ▶ **complement** $\bar{A} = B \setminus A$, where $A \subseteq B$ and B is the set of all considered objects (in a given context)



Tuples

- ▶ **k -tuple**: ordered sequence of k objects
- ▶ written (o_1, \dots, o_k) or $\langle o_1, \dots, o_k \rangle$
- ▶ unlike sets, **order matters** ($\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$)
- ▶ objects may occur multiple times in a tuple
- ▶ objects contained in tuples are called **components**
- ▶ terminology:
 - ▶ $k = 2$: (ordered) pair
 - ▶ $k = 3$: triple
- ▶ if k is clear from context (or does not matter), often just called **tuple**

Cartesian Product

- ▶ for sets M_1, M_2, \dots, M_n , the **Cartesian product** $M_1 \times \dots \times M_n$ is the set
 $M_1 \times \dots \times M_n = \{\langle o_1, \dots, o_n \rangle \mid o_1 \in M_1, \dots, o_n \in M_n\}$.
- ▶ Example: $M_1 = \{a, b, c\}, M_2 = \{1, 2\}$,
 $M_1 \times M_2 = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle\}$
- ▶ special case: $M^k = M \times \dots \times M$ (k times)
- ▶ example with $M = \{1, 2\}$:
 $M^2 = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}$

Relations

- ▶ an n -ary **relation** R over the sets M_1, \dots, M_n is a subset of their Cartesian product: $R \subseteq M_1 \times \dots \times M_n$.
- ▶ example with $M = \{1, 2\}$:
 $R_{\leq} \subseteq M^2$ as $R_{\leq} = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 2 \rangle\}$

A2.2 Functions

Functions

Definition (Total Function)

A (total) **function** $f : D \rightarrow C$ (with sets D, C) maps **every value** of its **domain** D to **exactly one value** of its **codomain** C .

Example

- ▶ $square : \mathbb{Z} \rightarrow \mathbb{Z}$ with $square(x) = x^2$
- ▶ $add : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ with $add(x, y) = x + y$
- ▶ $add_{\mathbb{R}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $add_{\mathbb{R}}(x, y) = x + y$

Functions: Example

Example

Let $Q = \{q_0, q_1, q_2, q_{\text{accept}}, q_{\text{reject}}\}$ and $\Gamma = \{0, 1, \square\}$.

Define $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ by

δ	0	1	\square
q_0	$\langle q_0, 0, R \rangle$	$\langle q_0, 1, R \rangle$	$\langle q_1, \square, L \rangle$
q_1	$\langle q_2, 1, L \rangle$	$\langle q_1, 0, L \rangle$	$\langle q_{\text{reject}}, 1, L \rangle$
q_2	$\langle q_2, 0, L \rangle$	$\langle q_2, 1, L \rangle$	$\langle q_{\text{accept}}, \square, R \rangle$

Then, e. g., $\delta(q_0, 1) = \langle q_0, 1, R \rangle$

Partial Functions

Definition (Partial Function)

A **partial function** $f : X \rightarrow_p Y$ maps every value in X to **at most** one value in Y .

If f does not map $x \in X$ to any value in Y , then f is **undefined** for x .

Example

$f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow_p \mathbb{N}_0$ with

$$f(x, y) = \begin{cases} x - y & \text{if } y \leq x \\ \text{undefined} & \text{otherwise} \end{cases}$$

A2.3 Summary

Summary

- ▶ **sets:** unordered, contain every element at most once
- ▶ **tuples:** ordered, can contain the same object multiple times
- ▶ **Cartesian product:** $M_1 \times \cdots \times M_n$ set of all n -tuples where the i -th component is in M_i
- ▶ **function** $f : X \rightarrow Y$ maps every value in X to exactly one value in Y
- ▶ **partial function** $g : X \rightarrow_p Y$ may be undefined for some values in X