Theory of Computer Science

B6. Regular Languages: Pumping Lemma

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Pumping Lemma: Motivation

You can show that
a language is regular by specifying
an appropriate grammar, finite
automaton, or regular expression.
How can you you show that a language
is not regular?

- Direct proof that no regular grammar exists that generates the language
  $\rightsquigarrow$ difficult in general
- Pumping lemma: use a necessary property that holds for all regular languages.

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Pumping Lemma

Theorem (Pumping Lemma)

If L is a regular language then there is a number $p \in \mathbb{N}$ (a pumping number for L) such that all words $x \in L$ with $|x| \geq p$ can be split into $x = uvw$ with the following properties:

1. $|v| \geq 1$,
2. $|uv| \leq p$, *and*
3. $uv^iw \in L$ *for all* $i = 0, 1, 2, \ldots$.

Question: what if L is finite?

Pumping Lemma: Proof

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1. $|v| \geq 1$,
2. $|uv| \leq p$, *and*
3. $uv^i w \in L$ *for all* $i = 0, 1, 2, \ldots$.

Proof.

For regular L there exists a DFA $M = \langle Q, \Sigma, \delta, q_0, E \rangle$ with $\mathcal{L}(M) = L$. We show that $p = |Q|$ has the desired properties.

Consider an arbitrary $x \in \mathcal{L}(M)$ with length $|x| \geq |Q|$. Including the start state, M visits $|x| + 1$ states while reading x. Because of $|x| \geq |Q|$ at least one state has to be visited twice. ...

Pumping Lemma: Proof

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1. $|v| \geq 1$,
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3. $uv^i w \in L$ *for all* $i = 0, 1, 2, \ldots$.

Proof (continued).

Choose a split $x = uvw$ so M is in the same state after reading u and after reading uv. Obviously, we can choose the split in a way that $|v| \geq 1$ and $|uv| \leq |Q|$ are satisfied. ...

Pumping Lemma: Proof

Theorem (Pumping Lemma)

If L is a regular language then there is a number $p \in \mathbb{N}$ (a pumping number for L) such that all words $x \in L$ with $|x| \geq p$ can be split into $x = uvw$ with the following properties:

1. $|v| \geq 1$,
2. $|uv| \leq p$, *and*
3. $uv^i w \in L$ *for all* $i = 0, 1, 2, \ldots$.

Proof (continued).

The word v corresponds to a loop in the DFA after reading u and can thus be repeated arbitrarily often. Every subsequent continuation with w ends in the same end state as reading x. Therefore $uv^i w \in \mathcal{L}(M) = L$ is satisfied for all $i = 0, 1, 2, \ldots$. □

Pumping Lemma: Application

Using the pumping lemma (PL):

Proof of Nonregularity

- If L is regular, then the pumping lemma holds for L.
- By contraposition: if the PL does not hold for L, then L cannot be regular.
- That is: if there is no $p \in \mathbb{N}$ with the properties of the PL, then L cannot be regular.

Pumping Lemma: Caveat

Caveat:

The pumping lemma is a necessary condition for a language to be regular, but not a sufficient one.

- $\rightsquigarrow$ there are languages that satisfy the pumping lemma conditions but are not regular
- $\rightsquigarrow$ for such languages, other methods are needed to show that they are not regular (e.g., the Myhill-Nerode theorem)

Pumping Lemma: Example

Example

The language $L = \{\mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\}$ is not regular.

Proof.

Assume L is regular. Then let p be a pumping number for L.

The word $x = \mathtt{a}^p\mathtt{b}^p$ is in L and has length $\geq p$.
Let $x = uvw$ be a split with the properties of the PL.

Then the word $x' = uv^2w$ is also in L. Since $|uv| \leq p$, uv consists only of symbols a and $x' = \mathtt{a}^{|u|}\mathtt{a}^{2|v|}\mathtt{a}^{p-|uv|}\mathtt{b}^p = \mathtt{a}^{p+|v|}\mathtt{b}^p$.

Since $|v| \geq 1$ it follows that $p + |v| \neq p$ and thus $x' \notin L$.

This is a contradiction to the PL. $\rightsquigarrow$ L is not regular. $\square$

Pumping Lemma: Another Example I

Example

The language $L = \{\mathtt{a}\mathtt{b}^n\mathtt{a}\mathtt{c}^{n+2} \mid n \in \mathbb{N}\}$ is not regular.

Proof.

Assume L is regular. Then let p be a pumping number for L.

The word $x = \mathtt{a}\mathtt{b}^p\mathtt{a}\mathtt{c}^{p+2}$ is in L and has length $\geq p$.
Let $x = uvw$ be a split with the properties of the PL.

From $|uv| \leq p$ and $|v| \geq 1$ we know that uv consists of one $\mathtt{a}$ followed by at most $p - 1$ $\mathtt{b}$s.

We distinguish two cases, $|u| = 0$ and $|u| > 0$. …

Pumping Lemma: Another Example II

Example

The language $L = \{\mathtt{ab}^n\mathtt{ac}^{n+2} \mid n \in \mathbb{N}\}$ is not regular.

Proof (continued).

If $|u| = 0$, then word v starts with an $\mathtt{a}$.
Hence, $uv^0w = \mathtt{b}^{p-|v|+1}\mathtt{ac}^{p+2}$ does not start with symbol $\mathtt{a}$ and is therefore not in L. This is a contradiction to the PL.

If $|u| > 0$, then word v consists only of $\mathtt{b}$s.
Consider $uv^0w = \mathtt{ab}^{p-|v|}\mathtt{ac}^{p+2}$. As $|v| \geq 1$, this word does not contain two more $\mathtt{c}$s than $\mathtt{b}$s and is therefore not in language L. This is a contradiction to the PL.

We have in all cases a contradiction to the PL.
$\rightsquigarrow$ L is not regular. $\square$

Pumping Lemma: Exercise

This was an exam question in 2020:

Use the pumping lemma to prove that
$L = \{a^m b^n \mid m \geq 0, n < m\}$ is not regular.

Summary

- The pumping lemma can be used to show that a language is not regular.