

Theory of Computer Science

C6. Rice's Theorem

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Rice's Theorem

Rice's Theorem (1)

- We have shown that a number of problems are undecidable:
 - halting problem H
 - halting problem on empty tape H_0
 - post correspondence problem PCP
- Many more results of this type could be shown.
- Instead, we prove a much more general result, **Rice's theorem**, which shows that a very large class of different problems are undecidable.
- Rice's theorem can be summarized informally as:
every non-trivial question about **what** a given Turing machine computes is undecidable.

Rice's Theorem (2)

Theorem (Rice's Theorem)

Let $\mathcal{R}$ be the class of all computable functions.

*Let $\mathcal{S}$ be an **arbitrary** subset of $\mathcal{R}$ except $\mathcal{S} = \emptyset$ or $\mathcal{S} = \mathcal{R}$.*

Then the language

$$C(\mathcal{S}) = \{w \in \{0, 1\}^* \mid \text{the function computed by } M_w \text{ is in } \mathcal{S}\}$$

is undecidable.

German: Satz von Rice

Question: why the restriction to $\mathcal{S} \neq \emptyset$ and $\mathcal{S} \neq \mathcal{R}$?

Extension (without proof): in most cases neither $C(\mathcal{S})$ nor $\overline{C(\mathcal{S})}$ is Turing-recognizable. (But there are sets $\mathcal{S}$ for which one of the two languages is Turing-recognizable.)

Rice's Theorem (3)

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Let Q be a Turing machine that computes q .

...

Rice's Theorem (4)

Proof (continued).

We show that $\bar{H}_0 \leq C(\mathcal{S})$.

Consider function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$,
where $f(w)$ is defined as follows:

- Construct TM M that first behaves on input y like M_w on the empty tape (independently of what y is).
- Afterwards (if that computation terminates!) M clears the tape, creates the start configuration of Q for input y and then simulates Q .
- $f(w)$ is the encoding of this TM M

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f is total and computable.

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For all words $w \in \{0, 1\}^*$:

- $w \in H_0 \implies M_w \text{ terminates on } \varepsilon$
- $\implies M_{f(w)} \text{ computes the function } q$
- $\implies \text{the function computed by } M_{f(w)} \text{ is not in } \mathcal{S}$
- $\implies f(w) \notin C(\mathcal{S})$

...

Rice's Theorem (6)

Proof (continued).

Further:

- $w \notin H_0 \implies M_w$ does not terminate on ε
- $\implies M_{f(w)}$ computes the function Ω
- $\implies$ the function computed by $M_{f(w)}$ is in $\mathcal{S}$
- $\implies f(w) \in C(\mathcal{S})$

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$$\begin{aligned} w \notin H_0 &\implies M_w \text{ does not terminate on } \varepsilon \\ &\implies M_{f(w)} \text{ computes the function } \Omega \\ &\implies \text{the function computed by } M_{f(w)} \text{ is in } \mathcal{S} \\ &\implies f(w) \in C(\mathcal{S}) \end{aligned}$$

Together this means: $w \notin H_0$ iff $f(w) \in C(\mathcal{S})$,
thus $w \in \bar{H}_0$ iff $f(w) \in C(\mathcal{S})$.

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Together this means: $w \notin H_0$ iff $f(w) \in C(\mathcal{S})$,
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We can conclude that $C(\mathcal{S})$ is undecidable.

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Rice's Theorem (7)

Proof (continued).

Case 2: $\Omega \notin \mathcal{S}$

Analogous to Case 1 but this time choose $q \in \mathcal{S}$.

The corresponding function f then reduces H_0 to $C(\mathcal{S})$.

Thus, it also follows in this case that $C(\mathcal{S})$ is undecidable. □

Rice's Theorem: Consequences

Was it worth it?

We can now conclude immediately that (for example) the following informally specified problems are all undecidable:

- Does a given TM compute a constant function?
- Does a given TM compute a total function (i. e. will it always terminate, and in particular terminate in a “correct” configuration)?
- Is the output of a given TM always longer than its input?
- Does a given TM compute the identity function?
- Does a given TM compute the computable function f ?
- ...

Rice's Theorem: Examples

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 $\mathcal{S} = \{f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 \mid f(x, y) = x + y\}$

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- Does a given TM compute the computable function f ?
 $\mathcal{S} = \{f\}$
(full automatization of software verification is impossible)

Exercise

This was an exam question in 2019.

Is the following informally described problem decidable? Give a brief justification.

Given a deterministic Turing machine M , is the language recognized by M regular?



Rice's Theorem: Pitfalls

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- Show that $\{w \mid M_w \text{ traverses all states on every input}\}$ is undecidable.

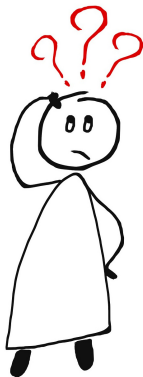
Rice's theorem not directly applicable because not a semantic property (the function computed by M_w can also be computed by a TM that does not traverse all states)

Rice's Theorem: Practical Applications

Undecidable due to Rice's theorem + a small reduction:

- **automated debugging:**
 - Can a given variable ever receive a null value?
 - Can a given assertion in a program ever trigger?
 - Can a given buffer ever overflow?
- **virus scanners and other software security analysis:**
 - Can this code do something harmful?
 - Is this program vulnerable to SQL injections?
 - Can this program lead to a privilege escalation?
- **optimizing compilers:**
 - Is this dead code?
 - Is this a constant expression?
 - Can pointer aliasing happen here?
 - Is it safe to parallelize this code path?
- **parallel program analysis:**
 - Is a deadlock possible here?
 - Can a race condition happen here?

Questions



Questions?

Further Undecidable Problems

And What Else?

- Here we conclude our discussion of undecidable problems.
- Many more undecidable problems exist.
- In this section, we briefly discuss some further classical results.

Undecidable Grammar Problems

Some Grammar Problems

Given context-free grammars G_1 and G_2 , ...

- ... is $\mathcal{L}(G_1) \cap \mathcal{L}(G_2) = \emptyset$?
- ... is $|\mathcal{L}(G_1) \cap \mathcal{L}(G_2)| = \infty$?
- ... is $\mathcal{L}(G_1) \cap \mathcal{L}(G_2)$ context-free?
- ... is $\mathcal{L}(G_1) \subseteq \mathcal{L}(G_2)$?
- ... is $\mathcal{L}(G_1) = \mathcal{L}(G_2)$?

Given a context-sensitive grammar G , ...

- ... is $\mathcal{L}(G) = \emptyset$?
- ... is $|\mathcal{L}(G)| = \infty$?

$\rightsquigarrow$ all undecidable by reduction from PCP
(see Schöning, Chapter 2.8)

Gödel's First Incompleteness Theorem (1)

Definition (Arithmetic Formula)

An **arithmetic formula** is a closed predicate logic formula using

- constant symbols 0 and 1,
- function symbols $+$ and $\cdot$, and
- equality ($=$) as the only relation symbols.

It is called **true** if it is true under the usual interpretation of 0, 1, $+$ and $\cdot$ over $\mathbb{N}_0$.

German: arithmetische Formel

Beispiel: $\forall x \exists y \forall z (((x \cdot y) = z) \wedge ((1 + x) = (x \cdot y)))$

Gödel's First Incompleteness Theorem (2)

Gödel's First Incompleteness Theorem

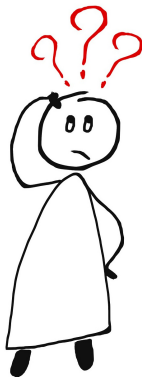
The problem of **deciding if a given arithmetic formula is true** is undecidable.

Moreover, neither it nor its complement are Turing-recognizable.

As a consequence, there exists no sound and complete proof system for arithmetic formulas.

German: erster Gödelscher Unvollständigkeitssatz

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→ easiest with [Rice' theorem](#)
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- other problems
 - [directly with the definition of undecidability](#)
→ usually quite complicated
 - [reduction from an undecidable problem](#), e.g.
→ halting problem (H)
→ Post correspondence problem (PCP)

What's Next?

contents of this course:

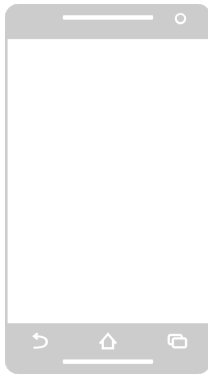
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Quiz



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