

Theory of Computer Science

C5. Post Correspondence Problem

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C5.1 Post Correspondence Problem

C5.2 (Un-)Decidability of PCP

C5.1 Post Correspondence Problem

More Options for Reduction Proofs?

- ▶ We can prove the undecidability of a problem with a reduction from an undecidable problem.
- ▶ The [halting problem](#) and the [halting problem on the empty tape](#) are possible options for this.
- ▶ both halting problem variants are quite similar 😊

→ We want a wider selection for reduction proofs
→ Is there some problem that is different in flavor?

Post correspondence problem
(named after mathematician [Emil Leon Post](#))

Post Correspondence Problem: Example

Example (Post Correspondence Problem)

Given: different kinds of "dominos"

1: <table border="1" style="display: inline-table; vertical-align: middle;"><tr><td>1</td></tr><tr><td>101</td></tr></table>	1	101	2: <table border="1" style="display: inline-table; vertical-align: middle;"><tr><td>10</td></tr><tr><td>00</td></tr></table>	10	00	3: <table border="1" style="display: inline-table; vertical-align: middle;"><tr><td>011</td></tr><tr><td>11</td></tr></table>	011	11
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(an infinite number of each kind)

Question: Is there a sequence of dominos such that the upper and lower row match (= are equal)

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Post Correspondence Problem: Definition

Definition (Post Correspondence Problem PCP)

Given: Finite sequence of pairs of words $(t_1, b_1), (t_2, b_2), \dots, (t_k, b_k)$, where $t_i, b_i \in \Sigma^+$ (for an arbitrary alphabet Σ)

Question: Is there a sequence

$i_1, i_2, \dots, i_n \in \{1, \dots, k\}$, $n \geq 1$,
with $t_{i_1} t_{i_2} \dots t_{i_n} = b_{i_1} b_{i_2} \dots b_{i_n}$?

A solution of the correspondence problem is such a sequence i_1, \dots, i_n , which we call a match.

Exercise (slido)

Consider PCP instance $(11, 1), (0, 00), (10, 01), (01, 11)$.



Is 2, 4, 3, 3, 1 a match?

Given-Question Form vs. Definition as Set

So far: problems defined as sets

Now: definition in Given-Question form

Definition (new problem P)

Given: Instance \mathcal{I}

Question: Does \mathcal{I} have a specific property?

corresponds to definitions

Definition (new problem P)

The problem P is the language

$P = \{w \mid w \text{ encodes an instance } \mathcal{I} \text{ with the required property}\}$.

Definition (new problem P)

The problem P is the language

$P = \{\langle \mathcal{I} \rangle \mid \mathcal{I} \text{ is an instance with the required property}\}$.

PCP Definition as Set

We can alternatively define PCP as follows:

Definition (Post Correspondence Problem PCP)

The Post Correspondence Problem PCP is the set

$\text{PCP} = \{w \mid w \text{ encodes a sequence of pairs of words } (t_1, b_1), (t_2, b_2), \dots, (t_k, b_k), \text{ for which there is a sequence } i_1, i_2, \dots, i_n \in \{1, \dots, k\} \text{ such that } t_{i_1} t_{i_2} \dots t_{i_n} = b_{i_1} b_{i_2} \dots b_{i_n}\}.$

C5.2 (Un-)Decidability of PCP

Post Correspondence Problem

PCP cannot be so hard, huh?

– Is it?

1101	0110	1
1	11	110

Formally: $K = ((1101, 1), (0110, 11), (1, 110))$
 \rightarrow Shortest match has length 252!

10	0	100
0	001	1

Formally: $K = ((10, 0), (0, 001), (100, 1))$
 \rightarrow Unsolvable

PCP: Turing-recognizability

Theorem (Turing-recognizability of PCP)

PCP is *Turing-recognizable*.

Proof.

Recognition procedure for input w :

- ▶ If w encodes a sequence $(t_1, b_1), \dots, (t_k, b_k)$ of pairs of words:
 Test systematically longer and longer sequences i_1, i_2, \dots, i_n whether they represent a match.
 If yes, terminate and return "yes".
- ▶ If w does not encode such a sequence: enter an infinite loop.

If $w \in \text{PCP}$ then the procedure terminates with "yes",
 otherwise it does not terminate. □

PCP: Undecidability

Theorem (Undecidability of PCP)

PCP is *undecidable*.

Proof via an intermediate other problem

modified PCP (MPCP)

- ➊ Reduce MPCP to PCP ($\text{MPCP} \leq \text{PCP}$)
- ➋ Reduce halting problem to MPCP ($H \leq \text{MPCP}$)

→ Let's get started...

MPCP: Definition

Definition (Modified Post Correspondence Problem MPCP)

Given: Sequence of word pairs as for PCP

Question: Is there a match $i_1, i_2, \dots, i_n \in \{1, \dots, k\}$ with $i_1 = 1$?

Reducibility of MPCP to PCP(1)

Lemma

$\text{MPCP} \leq \text{PCP}$.

Proof.

Let $\#, \$ \notin \Sigma$. For word $w = a_1 a_2 \dots a_m \in \Sigma^+$ define

$$\begin{aligned}\bar{w} &= \# a_1 \# a_2 \# \dots \# a_m \# \\ \grave{w} &= \# a_1 \# a_2 \# \dots \# a_m \\ \acute{w} &= a_1 \# a_2 \# \dots \# a_m \#\end{aligned}$$

For input $C = ((t_1, b_1), \dots, (t_k, b_k))$ define

$$f(C) = ((\bar{t}_1, \grave{b}_1), (\acute{t}_1, \grave{b}_1), (\acute{t}_2, \grave{b}_2), \dots, (\acute{t}_k, \grave{b}_k), (\$, \$\$))$$

...

Reducibility of MPCP to PCP(2)

Proof (continued).

$$f(C) = ((\bar{t}_1, \grave{b}_1), (\acute{t}_1, \grave{b}_1), (\acute{t}_2, \grave{b}_2), \dots, (\acute{t}_k, \grave{b}_k), (\$, \$\$))$$

Function f is *computable*, and can suitably get extended to a *total* function. It holds that

C has a solution with $i_1 = 1$ iff $f(C)$ has a solution:

Let $1, i_2, i_3, \dots, i_n$ be a solution for C . Then $1, i_2 + 1, \dots, i_n + 1, k + 2$ is a solution for $f(C)$.

If i_1, \dots, i_n is a match for $f(C)$, then (due to the construction of the word pairs) there is a $m \leq n$ such that $i_1 = 1, i_m = k + 2$ and $i_j \in \{2, \dots, k + 1\}$ for $j \in \{2, \dots, m - 1\}$. Then $1, i_2 - 1, \dots, i_{m-1} - 1$ is a solution for C .

⇒ f is a reduction from MPCP to PCP. □

PCP: Undecidability – Where are we?

Theorem (Undecidability of PCP)

PCP is *undecidable*.

Proof via an intermediate other problem

modified PCP (MPCP)

- ➊ Reduce MPCP to PCP ($\text{MPCP} \leq \text{PCP}$) ✓
- ➋ Reduce halting problem to MPCP ($H \leq \text{MPCP}$)

Reducibility of H to MPCP(1)

Lemma

$H \leq \text{MPCP}$.

Proof.

Goal: Construct for Turing machine

$M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ and word $w \in \Sigma^*$ an MPCP instance $C = ((t_1, b_1), \dots, (t_k, b_k))$ such that

M started on w terminates iff $C \in \text{MPCP}$.

...

Reducibility of H to MPCP(2)

Proof (continued).

Idea:

- ▶ Sequence of words describes sequence of configurations of the TM
- ▶ “ t -row” follows “ b -row” $x : \langle \# c_0 \# c_1 \# c_2 \# \rangle$
 $y : \langle \# c_0 \# c_1 \# c_2 \# c_3 \# \rangle$
- ▶ Configurations get mostly just copied, only the area around the head changes.
- ▶ After a terminating configuration has been reached: make row equal by deleting the configuration.

...

Reducibility of H to MPCP(3)

Proof (continued).

Alphabet of C is $\Gamma \cup Q \cup \{\#\}$.

1. Pair: $(\#, \#q_0 w \#)$

Other pairs:

- ➊ copy: (a, a) for all $a \in \Gamma \cup \{\#\}$
- ➋ transition:

(qa, cq') if $\delta(q, a) = (q', c, R)$
 $(q\#, cq'\#)$ if $\delta(q, \square) = (q', c, R)$

...

Reducibility of H to MPCP(4)

Proof (continued).

- ($bqa, q'bc$) if $\delta(q, a) = (q', c, L)$ for all $b \in \Gamma$
- ($bq\#, q'bc\#$) if $\delta(q, \square) = (q', c, L)$ for all $b \in \Gamma$
- ($\#qa, \#q'c$) if $\delta(q, a) = (q', c, L)$
- ($\#q\#, \#q'c\#$) if $\delta(q, \square) = (q', c, L)$

- ③ deletion: (aq, q) and (qa, q)
for all $a \in \Gamma$ and $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$
- ④ finish: $(q\#\#, \#)$ for all $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$

...

Reducibility of H to MPCP(5)

Proof (continued).

“ \Rightarrow ” If M terminates on input w , there is a sequence c_0, \dots, c_t of configurations with

- ▶ $c_0 = q_0 w$ is the start configuration
- ▶ c_t is a terminating configuration
- ($c_t = uqv$ mit $u, v \in \Gamma^*$ and $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$)
- ▶ $c_i \vdash c_{i+1}$ for $i = 0, 1, \dots, t-1$

Then C has a match with the overall word

$\#c_0\#c_1\#\dots\#c_t\#c'_t\#c''_t\#\dots\#q_e\#\#$

Up to c_t : “ t -row” follows “ b -row”

From c'_t : deletion of symbols adjacent to terminating state. ...

Reducibility of H to MPCP(6)

Proof (continued).

“ \Leftarrow ” If C has a solution, it has the form

$\#c_0\#c_1\#\dots\#c_n\#\#$,

with $c_0 = q_0 w$. Moreover, there is an $\ell \leq n$, such that q_{accept} or q_{reject} occurs for the first time in c_ℓ .

All c_i for $i \leq \ell$ are configurations of M and $c_i \vdash c_{i+1}$ for $i \in \{0, \dots, \ell-1\}$.

c_0, \dots, c_ℓ is hence the sequence of configurations of M on input w , which shows that the TM terminates. \square

PCP: Undecidability – Done!

Theorem (Undecidability of PCP)

PCP is *undecidable*.

Proof via an intermediate other problem
modified PCP (MPCP)

- ① Reduce MPCP to PCP ($\text{MPCP} \leq \text{PCP}$) \checkmark
- ② Reduce halting problem to MPCP ($H \leq \text{MPCP}$) \checkmark

Proof.

Due to $H \leq \text{MPCP}$ and $\text{MPCP} \leq \text{PCP}$ it holds that $H \leq \text{PCP}$.
Since H is undecidable, also PCP must be undecidable. \square

Summary

- ▶ Post Correspondence Problem:
Find a sequence of word pairs s.t. the concatenation of all first components equals the one of all second components.
- ▶ The Post Correspondence Problem is **Turing-recognizable** but **not decidable**.