

Theory of Computer Science

C2. The Halting Problem

Gabriele Röger

University of Basel

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Turing-recognizable vs. decidable

Plan for this Chapter

- We will first revisit the notions **Turing-recognizable** and **Turing-decidable** and identify a connection between the two concepts.
- Then we will get to know an important undecidable problem, the **halting problem**.
- We show that it **is Turing-recognizable**...
- ... but **not Turing-decidable**.
- From these results we can conclude that **there are languages that are not Turing-recognizable**.
- Some of the postponed results on the closure and decidability properties of type 0 languages are direct implications our findings.

Reminder: Turing-recognizable and Turing-decidable

Definition (Turing-recognizable Language)

We call a language **Turing-recognizable** if some deterministic Turing machine recognizes it.

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We call a language **Turing-recognizable** if some deterministic Turing machine recognizes it.

A Turing machine that halts on all inputs (entering q_{reject} or q_{accept}) is a **decider**. A decider that recognizes some language also is said to **decide** the language.

Definition (Turing-decidable Language)

We call a language **Turing-decidable** (or **decidable**) if some deterministic Turing machine decides it.

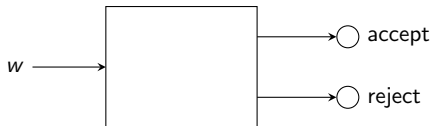
Intuition

Are these two definitions meaningfully different?

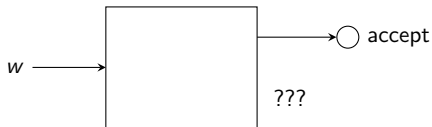
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(Turing-)decidable:



Turing-recognizable

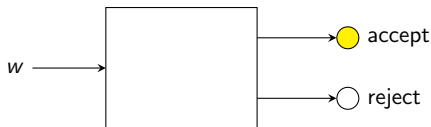


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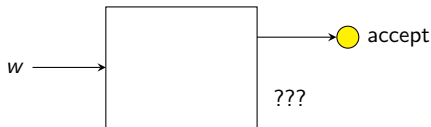
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Case 1: $w \in L$

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Turing-recognizable

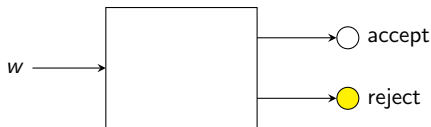


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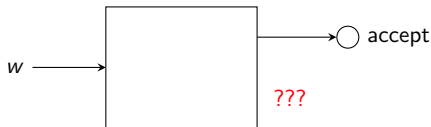
Are these two definitions meaningfully different? Yes!

Case 2: $w \notin L$

(Turing-)decidable:



Turing-recognizable



Connection Turing-recognizable and Turing-decidable (1)

Reminder: For language L , we write \bar{L} to denote its complement.

Theorem (Decidable vs. Turing-recognizable)

A language L is decidable iff both L and \bar{L} are Turing-recognizable.

Proof.

(\Rightarrow) : obvious (**Why?**)

...

Connection Turing-recognizable and Turing-decidable (2)

Proof (continued).

(\Leftarrow): Let M_L be a DTM that recognizes L ,
and let $M_{\bar{L}}$ be a DTM that recognizes \bar{L} .

The following algorithm decides L :

On a given input word w proceed as follows:

FOR $s := 1, 2, 3, \dots$:

IF M_L stops on w in s steps in the accept state:

ACCEPT

IF $M_{\bar{L}}$ stops on w in s steps in the accept state:

REJECT



Connection Turing-recognizable and Turing-decidable (2)

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Why don't we first entirely simulate M_L on the input
and only afterwards $M_{\bar{L}}$?

Example: Decidable \neq Known Algorithm

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- $L = \{n \in \mathbb{N} \mid \text{there are } n \text{ consecutive } 7\text{s}$
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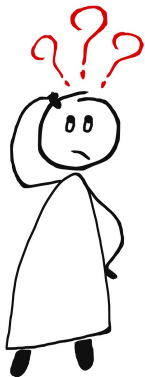
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- In both cases, we can decide the language.
- We just do not know what is the correct version
(and what is n_0 in case 2).

Questions



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The Halting Problem H

Reminder: Encodings of Turing Machines

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$$M_w = \begin{cases} M' & \text{if } w \text{ is the encoding of some DTM } M' \\ \hat{M} & \text{otherwise} \end{cases}$$

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- $M_w =$ “Turing machine encoded by w ”

Halting Problem

Definition (Halting Problem)

The **halting problem** is the language

$$H = \{w\#x \in \{0, 1, \#\}^* \mid w, x \in \{0, 1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

“Does the computation of the TM encoded by w halt on input x ?”

“Does a given piece of code terminate on a given input?”

The Halting Problem is Turing-recognizable

Theorem

The halting problem H is Turing-recognizable.

The following Turing machine U recognizes language H :

On input $w\#x$:

- 1 If the input contains more than one $\#$ then reject.
- 2 Simulate M_w (the TM encoded by w) on input x .
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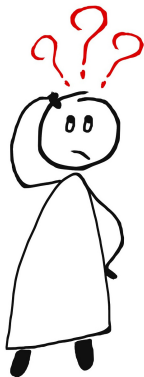
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U is an example of a so-called *universal Turing machine* which can simulate any other Turing machine from the description of that machine.

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H is Undecidable

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- Intuitively, this means that for this problem there is no algorithm that is correct and terminates on all inputs.
- To establish the undeciability of the halting problem, we will consider a situation where we run a Turing machine/algorithm on its own encoding/source code.
- We have seen something similar in the very first lecture. . .

Consider functions whose inputs are strings:

```
def program_returns_true_on_input(prog_code, input_str):  
    ...  
    # returns True if prog_code run on input_str returns True  
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def odd_program(prog_code):  
    if program_returns_true_on_input(prog_code, prog_code):  
        return False  
    else:  
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What is the return value of `odd_program`
if we run it on its own source code?

Consider two cases (if statement):

Case 1. `odd_program` returns True on its source code.

Case 2. `odd_program` returns False on its source code.



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- Overall, we have proven that there cannot be a program with the behaviour described by the comments.
- For the undecidability of the halting problem, we will use an analogous argument, only with Turing machines instead of code and termination instead of return values.

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Theorem (Undecidability of the Halting Problem)

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Proof.

Proof by contradiction: we assume that the halting problem H was decidable and derive a contradiction.

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Proof.

Proof by contradiction: we assume that the halting problem H was decidable and derive a contradiction.

So assume H is decidable and let D be a DTM that decides it. ...

Undecidability of the Halting Problem (2)

Proof (continued).

Construct the following new machine M that takes a word $x \in \{0, 1\}^*$ as input:

- 1 Execute D on the input $x\#x$.
- 2 If it rejects: accept.
- 3 Otherwise: enter an endless loop.

Undecidability of the Halting Problem (2)

Proof (continued).

Construct the following new machine M that takes a word $x \in \{0, 1\}^*$ as input:

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Let w be the encoding of M . **How will M behave on input w ?**

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M run on w stops

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Contradiction! DTM M cannot exist.

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Contradiction! DTM M cannot exist.

\Rightarrow DTM D cannot exist, thus H is not decidable. □

A Language that is not Turing-recognizable

We have the following results:

- A language L is decidable iff both L and \bar{L} are Turing-recognizable.
- The halting problem H is Turing-recognizable but not decidable.

Corollary

*The complement \bar{H} of the halting problem H is **not Turing-recognizable**.*

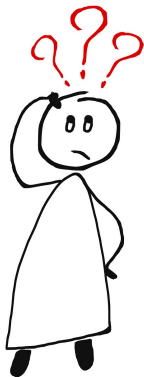
Exercises

- True or false? There is a grammar that generates H .
- True or false? Not all languages are of type 0.

Justify your answers.



Questions



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Reprise: Type-0 Languages

Back to Chapter B11: Closure Properties

	Intersection	Union	Complement	Concatenation	Star
Type 3	Yes	Yes	Yes	Yes	Yes
Type 2	No	Yes	No	Yes	Yes
Type 1	Yes ⁽²⁾	Yes ⁽¹⁾	Yes ⁽²⁾	Yes ⁽¹⁾	Yes ⁽¹⁾
Type 0	Yes ⁽²⁾	Yes ⁽¹⁾	No ⁽³⁾	Yes ⁽¹⁾	Yes ⁽¹⁾

Proofs?

(1) proof via grammars, similar to context-free cases

(2) without proof

(3) proof in later chapters (part C)

Back to Chapter B11: Decidability

	Word problem	Emptiness problem	Equivalence problem	Intersection problem
Type 3	Yes	Yes	Yes	Yes
Type 2	Yes	Yes	No	No
Type 1	Yes ⁽¹⁾	No ⁽³⁾	No ⁽²⁾	No ⁽²⁾
Type 0	No ⁽⁴⁾	No ⁽⁴⁾	No ⁽⁴⁾	No ⁽⁴⁾

Proofs?

- (1) same argument we used for context-free languages
- (2) because already undecidable for context-free languages
- (3) without proof
- (4) proofs in later chapters (part C)

Answers to Old Questions

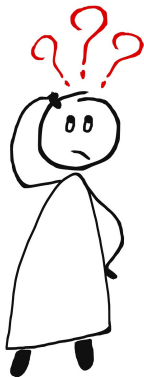
Closure properties:

- H is Turing-recognizable (and thus type 0) but not decidable.
- ↪ \bar{H} is **not** Turing-recognizable, thus **not** type 0.
- ↪ Type-0 languages are **not** closed under complement.

Decidability:

- H is type 0 but not decidable.
- ↪ **word problem** for type-0 languages not decidable
- ↪ emptiness, equivalence, intersection problem: **later in exercises**
(We are still missing some important results for this.)

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Summary

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- A language L is **decidable** iff both L and \bar{L} are **Turing-recognizable**.
- The **halting problem** is the language

$$H = \{w\#x \in \{0, 1, \#\}^* \mid w, x \in \{0, 1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

- The halting problem is **Turing-recognizable** but **undecidable**.
- The complement language \bar{H} is an example of a language that is **not even Turing-recognizable**.