Theory of Computer Science
B11. Type-1 and Type-0 Languages: Closure & Decidability

Gabriele Röger

University of Basel

April 12, 2021

B11.1 Turing Machines vs. Grammars

B11.2 Closure Properties and Decidability

B11.1 Turing Machines vs. Grammars

Turing Machines

We have seen several variants of Turing machines:

- ▶ Deterministic TM with head movements left or right
- ▶ Deterministic TM with head movements left, right or neutral
- ▶ Multitape Turing machines
- ▶ Nondeterministic Turing machines

All variants recognize the same languages.

We mentioned earlier that we can relate Turing machines to the Type-1 and Type-0 languages.

Reminder: Context-sensitive Grammar

Type-1 languages are also called context-sensitive languages.

Definition (Context-sensitive Grammar)

A context-sensitive grammar is a 4-tuple $\langle V, \Sigma, R, S \rangle$ with

- V finite set of variables (nonterminal symbols)
- Σ finite alphabet of terminal symbols with $V \cap \Sigma = \emptyset$
- $R \subseteq (V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ finite set of rules,
  where all rules are of the form $\alpha B \gamma \to \alpha \beta \gamma$
  with $B \in V$ and $\alpha, \gamma \in (V \cup \Sigma)^*$ and $\beta \in (V \cup \Sigma)^+$.
  Exception: $S \to \varepsilon$ is allowed if S never occurs on the
  right-hand side of a rule.
- $S \in V$ start variable.

One Automata Model for Two Grammar Types?

Picture courtesy of stockimages / FreeDigitalPhotos.net

Linear Bounded Automata: Idea

▶ Linear bounded automata are NTMs that may only use the part of the tape occupied by the input word.

▶ one way of formalizing this: NTMs where blank symbol may never be replaced by a different symbol

Linear Bounded Turing Machines: Definition

> **Definition (Linear Bounded Automata)**
> An NTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$
> is called a linear bounded automaton (LBA)
> if for all $q \in Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}$ and all transition rules
> $\langle q', c, y \rangle \in \delta(q, \square)$ we have $c = \square$.

German: linear beschränkte Turingmaschine

LBAs Recognize Type-1 Languages

Theorem
The languages that can be recognized by linear bounded automata are exactly the context-sensitive (type-1) languages.

Without proof.

proof sketch for grammar ⇒ NTM direction:

▶ computation of the NTM follows the production of the word in the grammar in opposite order

▶ accept when only the start symbol (and blanks) are left on the tape

▶ because the language is context-sensitive, we never need additional space on the tape (empty word needs special treatment)

NTMs Recognize Type-0 Languages

Theorem
The languages that can be recognized by nondeterministic Turing machines are exactly the type-0 languages.

Without proof.

proof sketch for grammar $\Rightarrow$ NTM direction:

▶ analogous to previous proof

▶ for grammar rules $w_1 \rightarrow w_2$ with $|w_1| > |w_2|$,
  we must "insert" symbols into the existing tape content;
  this is a bit tedious, but not very difficult

What about the Deterministic Variants?

We know that DTMs and NTMs recognize the same languages. Hence:

Corollary
The Turing-recognizable languages are exactly the Type-0 languages.

Note: It is an open problem whether deterministic LBAs can recognize exactly the type-1 languages.

B11.2 Closure Properties and Decidability

Closure Properties

	Intersection	Union	Complement	Concatenation	Star
Type 3	Yes	Yes	Yes	Yes	Yes
Type 2	No	Yes	No	Yes	Yes
Type 1	Yes[2]	Yes[1]	Yes[2]	Yes[1]	Yes[1]
Type 0	Yes[2]	Yes[1]	No[3]	Yes[1]	Yes[1]

Proofs?

(1) proof via grammars, similar to context-free cases

(2) without proof

(3) proof in later chapters (part C)

Decidability

	Word problem	Emptiness problem	Equivalence problem	Intersection problem
Type 3	Yes	Yes	Yes	Yes
Type 2	Yes	Yes	No	No
Type 1	Yes[1]	No[3]	No[2]	No[2]
Type 0	No[4]	No[4]	No[4]	No[4]

Proofs?

(1) same argument we used for context-free languages

(2) because already undecidable for context-free languages

(3) without proof

(4) proofs in later chapters (part C)

Summary

- Turing machines recognize exactly the type-0 languages.
- Linear bounded automata recognize exactly
  the context-sensitive languages.
- The context-sensitive and type-0 languages are closed
  under almost all usual operations.
  - exception: type-0 not closed under complement
- For context-sensitive and type-0 languages
  almost no problem is decidable.
  - exception: word problem for context-sensitive lang. decidable

What's Next?

contents of this course:

A. background ✓
   ▷ mathematical foundations and proof techniques

B. automata theory and formal languages ✓
   ▷ What is a computation?

C. Turing computability
   ▷ What can be computed at all?

D. complexity theory
   ▷ What can be computed efficiently?

E. more computability theory
   ▷ Other models of computability