# Theory of Computer Science

## B9. Turing Machines I

Gabriele Röger

University of Basel

April 7, 2021

# Turing Machines

# Automata for Type-1 and Type-0 Languages?

Finite automata recognize exactly the regular languages, push-down automata exactly the context-free languages. Are there automata models for context-sensitive and type-0 languages?

Picture courtesy of imagerymajestic / FreeDigitalPhotos.net

# Automata for Type-1 and Type-0 Languages?

Finite automata recognize exactly the regular languages, push-down automata exactly the context-free languages. Are there automata models for context-sensitive and type-0 languages?

Yes! ⇝ Turing machines

German: Turingmaschinen

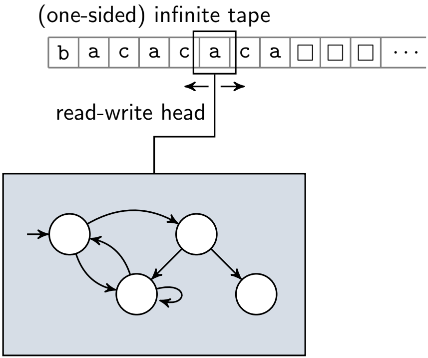
Picture courtesy of imagerymajestic / FreeDigitalPhotos.net

# Alan Turing (1912–1954)

Picture courtesy of Jon Callas / wikimedia commons

- British logician, mathematician, cryptanalyst and computer scientist
- most important work (for us): On Computable Numbers, with an Application to the Entscheidungsproblem
  $\rightsquigarrow$ Turing machines
- collaboration on Enigma decryption
- conviction due to homosexuality; pardoned by Elizabeth II in Dec. 2013
- Turing award most important science award in computer science

# Turing Machines: Conceptually



# Turing Machine: Definition

## Definition (Deterministic Turing Machine)

A (deterministic) Turing machine (DTM) is given by a 7-tuple $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$, where $Q, \Sigma, \Gamma$ are all finite sets and

- $Q$ is the set of states,
- $\Sigma$ is the input alphabet, not containing the blank symbol $\square$,
- $\Gamma$ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$,
- $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \to Q \times \Gamma \times \{\text{L}, \text{R}\})$ is the transition function,
- $q_0 \in Q$ is the start state,
- $q_{\text{accept}} \in Q$ is the accept state,
- $q_{\text{reject}} \in Q$ is the reject state, where $q_{\text{accept}} \neq q_{\text{reject}}$.

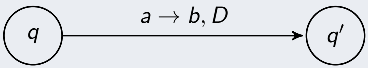
# Turing Machine: Transition Function

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ be a DTM.

## What is the Intuitive Meaning of the Transition Function $\delta$?

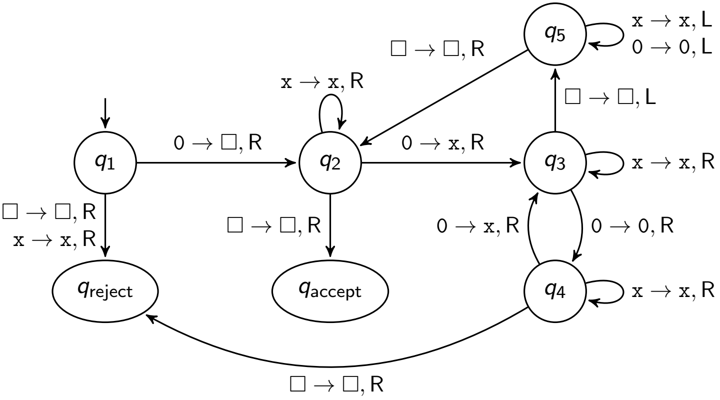
$\delta(q, a) = \langle q', b, D \rangle$:

- If $M$ is in state $q$ and reads $a$, then
- $M$ transitions to state $q'$ in the next step,
- replacing $a$ with $b$,
- and moving the head in direction $D \in \{\mathsf{L}, \mathsf{R}\}$, where:
  - R: one step to the right,
  - L: one step to the left, except if the head is on the left-most cell of the tape in which case there is no movement



# Deterministic Turing Machine: Example

$$\langle \{q_1, \ldots, q_5, q_{accept}, q_{reject}\}, \{0\}, \{0, \mathtt{x}, \square\}, \delta, q_1, q_{accept}, q_{reject} \rangle$$



# Turing Machine: Configuration

**Definition (Configuration of a Turing Machine)**

A configuration of a Turing machine
$M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$
is given by a triple $c \in \Gamma^* \times Q \times \Gamma^+$.

German: Konfiguration

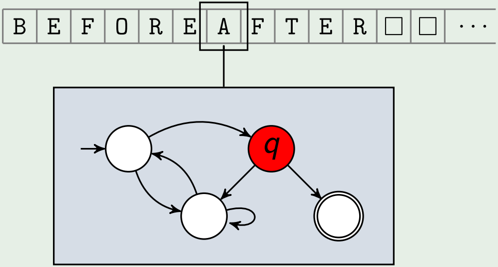
Configuration $\langle w_1, q, w_2 \rangle$ intuitively means that

- the non-empty or already visited part of the tape contains the word $w_1 w_2$,
- the read-write head is on the first symbol of $w_2$, and
- the TM is in state $q$.

# Turing Machine Configurations: Example

## Example

configuration $\langle \texttt{BEFORE}, q, \texttt{AFTER}\square\square \rangle$.



# Turing Machine Configurations: Start Configuration

Initially

- the TM is in start state $q_0$,
- the head is on the first tape cell, and
- the tape contains the input word $w$ followed by an infinite number of $\square$ entries.

The corresponding start configuration is $\langle \varepsilon, q_0, w \rangle$ if $w \neq \varepsilon$ and $\langle \varepsilon, q_0, \square \rangle$ if $w = \varepsilon$.

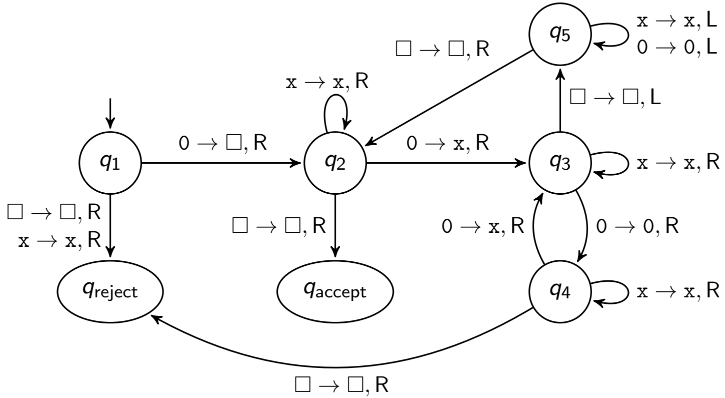
# Turing Machine: Step

## Definition (Transition/Step of a Turing Machine)

A DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ transitions from configuration $c$ to configuration $c'$ in one step ($c \vdash_M c'$) according to the following rules:

- $\langle a_1 \ldots a_m, q, b_1 \ldots b_n \rangle \vdash_M \langle a_1 \ldots a_{m-1}, q', a_m c b_2 \ldots b_n \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathrm{L} \rangle$, $m \geq 1$, $n \geq 1$
- $\langle \varepsilon, q, b_1 \ldots b_n \rangle \vdash_M \langle \varepsilon, q', c b_2 \ldots b_n \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathrm{L} \rangle$, $n \geq 1$
- $\langle a_1 \ldots a_m, q, b_1 \ldots b_n \rangle \vdash_M \langle a_1 \ldots a_m c, q', b_2 \ldots b_n \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathrm{R} \rangle$, $m \geq 0$, $n \geq 2$
- $\langle a_1 \ldots a_m, q, b_1 \rangle \vdash_M \langle a_1 \ldots a_m c, q', \square \rangle$
  if $\delta(q, b_1) = \langle q', c, \mathrm{R} \rangle$, $m \geq 0$

# Step: Exercise (Slido)




$\langle \square x, q_3, 00 \rangle \vdash ?$

# DTM: Accepted Words

Intuitively, a DTM accepts a word if its computation terminates in the accept state.

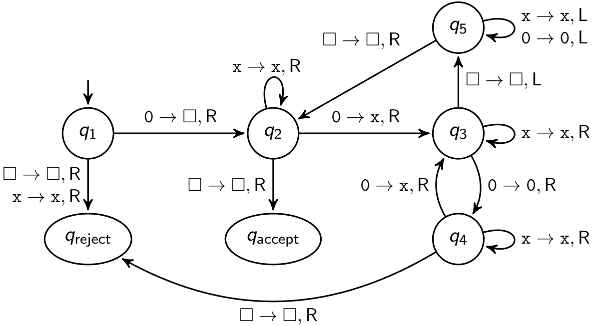
**Definition (Words Accepted by a DTM)**

DTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ accepts the word $w = a_1 \dots a_n$ if there is a sequence of configurations $c_0, \dots, c_k$ with

1. $c_0$ is the start configuration of $M$ on input $w$,
2. $c_i \vdash_M c_{i+1}$ for all $i \in \{0, \dots, k-1\}$, and
3. $c_k$ is an accepting configuration, i. e., a configuration with state $q_{\text{accept}}$.

# Accepted Word: Example

Does this Turing machine accept input 0000?




# DTM: Recognized Language

**Definition (Language Recognized by a DTM)**

Let $M$ be a deterministic Turing Machine
The language recognized by $M$ (or the language of $M$) is defined as $\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

# DTM: Recognized Language

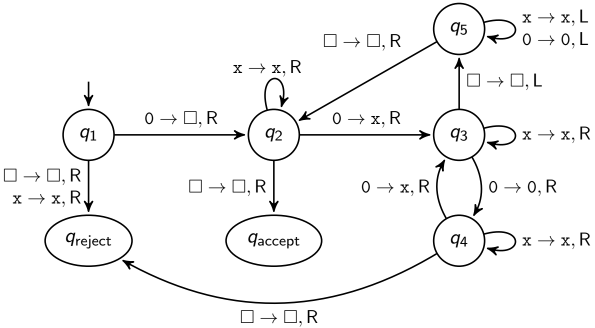
**Definition (Language Recognized by a DTM)**

Let $M$ be a deterministic Turing Machine
The language recognized by $M$ (or the language of $M$) is defined as $\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

**Definition (Turing-recognizable Language)**

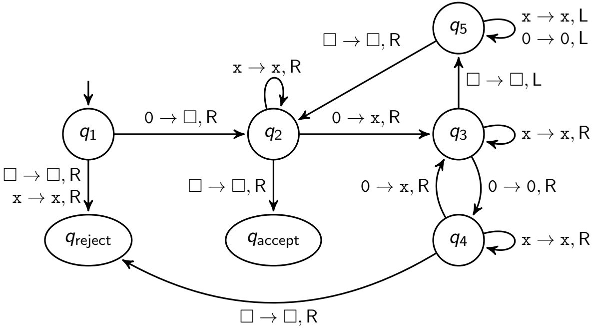
We call a language Turing-recognizable if some deterministic Turing machine recognizes it.

# Turing Machine: Example




1. Sweep left to right across the tape, sweeping off every other 0.
2. If in stage 1 the tape contained a single 0, accept.
3. If in stage 1 the tape contained more than one 0 and the number of 0s was odd, reject.
4. Return the head to the left end of the tape and go to stage 1.

# Recognized Language: Example




What language does the Turing machine recognize?

# Deciders

- A Turing machine either fails to accept an input
  - because it rejects it (entering $q_{\text{reject}}$) or
  - because it loops ($=$ does not halt).
- A Turing machine that halts on all inputs (entering $q_{\text{reject}}$ or $q_{\text{accept}}$) is called a decider.
- A decider that recognizes some language also is said to decide the language.

**Definition (Turing-decidable Language)**

We call a language Turing-decidable (or decidable) if some deterministic Turing machine decides it.

# Exercise

Specify the state diagram of a DTM that decides language

$$L = \{w\#w \mid w \in \{0,1\}^*\}.$$

# Summary

# Summary

- Turing machines only have finitely many states
  but an unbounded tape as "memory".
- Alan Turing proposed them as a mathematical model
  for arbitrary algorithmic computations.
- In this role, we will revisit them in the parts
  on computability and complexity theory.