Theory of Computer Science

B7. Context-free Languages: Normal Form and PDA

Gabriele Röger

University of Basel

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B7.1 Context-free Grammars

Repetition: Context-free Grammars

Definition (Context-free Grammar)

A context-free grammar is a 4-tuple $\langle V, \Sigma, R, S \rangle$ with

1. V finite set of variables,
2. Σ finite alphabet of terminal symbols (with $V \cap \Sigma = \emptyset$),
3. $R \subseteq V \times (V \cup \Sigma)^*$ finite set of rules,
4. $S \in V$ start variable.

Short-hand Notation for Rule Sets

We abbreviate several rules with the same left-hand side variable in a single line, using "|" for separating the right-hand sides.

For example, we write

$$\mathsf{X} \rightarrow \mathtt{0}\mathsf{Y}\mathtt{1} \mid \mathsf{XY}$$

for:

$$\mathsf{X} \rightarrow \mathtt{0}\mathsf{Y}\mathtt{1} \text{ and}$$
$$\mathsf{X} \rightarrow \mathsf{XY}$$

Context-free Grammars: Exercise

We have used the pumping lemma for regular languages to show that $L = \{\mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}_0\}$ is not regular.

Show that it is context-free by specifying a suitable grammar G with $\mathcal{L}(G) = L$.

B7.2 Chomsky Normal Form

Chomsky Normal Form: Motivation

As in other kinds of structured objects, normal forms for grammars are useful:

- they show which aspects are critical for defining grammars and which ones are just syntactic sugar
- they allow proofs and algorithms to be restricted to a limited set of grammars (inputs): those in normal form

Hence we now consider a normal form for context-free grammars.

Chomsky Normal Form: Definition

Definition (Chomsky Normal Form)

A context-free grammar G is in Chomsky normal form (CNF) if all rules have one of the following three forms:

- $A \rightarrow BC$ with variables A, B, C and B and C are not the start variable, or
- $A \rightarrow a$ with variable A and terminal symbol a, or
- $S \rightarrow \varepsilon$ with start variable S.

German: Chomsky-Normalform

formally: rule set $R \subseteq (V \times ((V \setminus \{S\})(V \setminus \{S\}) \cup \Sigma)) \cup \{\langle S, \varepsilon \rangle\}$

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof.

The following algorithm converts the rule set of $G = \langle V, \Sigma, R, S \rangle$ into CNF:

Step 1: Add new start variable S'.

Add a new variable S' which will be the start variable, and add a rule $S' \to S$, where S is the original start variable.

Afterwards, the (new) start variable does not occur on the right-hand side of a rule.

We will write V' for the new variable set ($V' = V \cup \{S'\}$) and R' for the new rule set. ...

Chomsky Normal Form: Theorem

Proof (continued).

Step 2: Eliminate ε-rules of the form $A \to \varepsilon$ ($A \neq S'$).

- Let V_ε be the set of variable from which one can derive the empty word. We find this set V_ε by first collecting all variables $A \in V'$ with rule $A \to \varepsilon \in R'$ and then successively adding additional variables B if there is a rule $B \to A_1 A_2 \dots A_k \in R'$ and the variables A_i are already in the set for all $1 \leq i \leq k$.
- Add rules that obviate the need for $A \to \varepsilon$ rules: for every existing rule $B \to w \in R'$ with $B \in V'$, $w \in (V' \cup \Sigma)^+$, let I_ε be the set of positions where w contains a variable $A \in V_\varepsilon$. For every non-empty set $I' \subseteq I_\varepsilon$, add a new rule $B \to w'$, where w' is constructed from w by removing the variables at all positions in I'.
- Remove all rules of the form $A \to \varepsilon$ ($A \neq S'$).

...

Step 2: Example

Consider $G = \langle\{\mathsf{X}, \mathsf{Y}, \mathsf{Z}, \mathsf{S}\}, \{\mathtt{a}, \mathtt{b}\}, R, S\rangle$ with rules:

$$\begin{aligned} \mathsf{S} &\to \varepsilon \mid \mathsf{XY} \\ \mathsf{X} &\to \mathtt{a}\mathsf{XY}\mathtt{b}\mathsf{X} \mid \mathsf{YZ} \mid \mathtt{ab} \\ \mathsf{Y} &\to \varepsilon \mid \mathtt{b} \\ \mathsf{Z} &\to \varepsilon \mid \mathtt{a} \end{aligned}$$

Chomsky Normal Form: Theorem

Proof (continued).

Step 3: Eliminate rules of the form $A \rightarrow B$ with variables A, B.

If there are sets of variables $\{B_1, \dots, B_k\}$ with rules
$B_1 \rightarrow B_2, B_2 \rightarrow B_3, \dots, B_{k-1} \rightarrow B_k, B_k \rightarrow B_1$,
then replace these variables by a new variable B.

We use V'' to denote the resulting set of variables.

Define a strict total order $<$ on the variables such that a rule $A \rightarrow B$ implies that $A < B$. Iterate from the largest to the smallest variable A and eliminate all rules of the form $A \rightarrow B$ while adding rules $A \rightarrow w$ for every rule $B \rightarrow w$ with $w \in (V'' \cup \Sigma)^+$. ...

Step 3: Example

Consider $G = \langle \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}, \mathsf{S}\}, \{\mathtt{a}, \mathtt{b}\}, R, S \rangle$ with rules:

$$\mathsf{S} \rightarrow \varepsilon \mid \mathsf{X}$$
$$\mathsf{X} \rightarrow \mathtt{a}\mathsf{Z}\mathtt{b}\mathsf{Y} \mid \mathsf{Y} \mid \mathtt{ab}$$
$$\mathsf{Y} \rightarrow \mathsf{Z} \mid \mathtt{b}$$
$$\mathsf{Z} \rightarrow \mathsf{Y} \mid \mathtt{b}\mathsf{X}\mathtt{a}$$

Chomsky Normal Form: Theorem

Proof (continued).

Step 4: Eliminate rules with terminal symbols on the right-hand side that do not have the form $A \rightarrow a$.

For every terminal symbol $a \in \Sigma$ add a new variable A_a and the rule $A_a \rightarrow a$.

Replace all terminal symbols in all rules that do not have the form $A \rightarrow a$ with the corresponding newly added variables. ...

Chomsky Normal Form: Theorem

Proof (continued).

Step 5: Eliminate rules of the form $A \rightarrow B_1 B_2 \dots B_k$ with $k > 2$

For every rule of the form $A \rightarrow B_1 B_2 \dots B_k$ with $k > 2$, add new variables $C_2, \dots, C_{k-1}$ and replace the rule with

$$\begin{aligned} A &\rightarrow B_1 C_2 \\ C_2 &\rightarrow B_2 C_3 \\ &\vdots \\ C_{k-1} &\rightarrow B_{k-1} B_k \end{aligned}$$

□

Chomsky Normal Form: Exercise

(Example taken from textbook by Sipser)

Consider $G = \langle \{\mathrm{A}, \mathrm{B}, \mathrm{S}\}, \{\mathtt{a}, \mathtt{b}\}, R, S \rangle$ with rules:

$$\mathrm{S} \to \mathrm{ASA} \mid \mathtt{a}\mathrm{B}$$
$$\mathrm{A} \to \mathrm{B} \mid \mathrm{S}$$
$$\mathrm{B} \to \varepsilon \mid \mathtt{b}$$

Specify a grammar G' in CNF with $\mathcal{L}(G') = \mathcal{L}(G)$.

Chomsky Normal Form: Length of Derivations

Observation

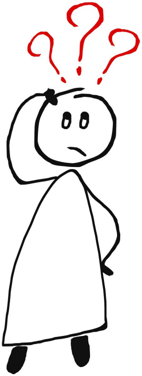
Let G be a grammar in Chomsky normal form,
and let $w \in \mathcal{L}(G)$ be a non-empty word generated by G.

Then all derivations of w have exactly $2|w| - 1$ derivation steps.

Proof.

$\rightsquigarrow$ Exercises □

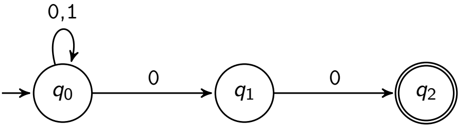
Questions



Questions?

B7.3 Push-Down Automata

Limitations of Finite Automata

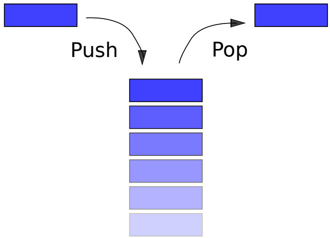


- Language L is regular.
  $\Longleftrightarrow$ There is a finite automaton that accepts L.
- What information can a finite automaton "store" about the already read part of the word?
- Infinite memory would be required for $L = \{x_1 x_2 \dots x_n x_n \dots x_2 x_1 \mid n > 0, x_i \in \{\mathtt{a}, \mathtt{b}\}\}$.
- therefore: extension of the automata model with memory

Stack

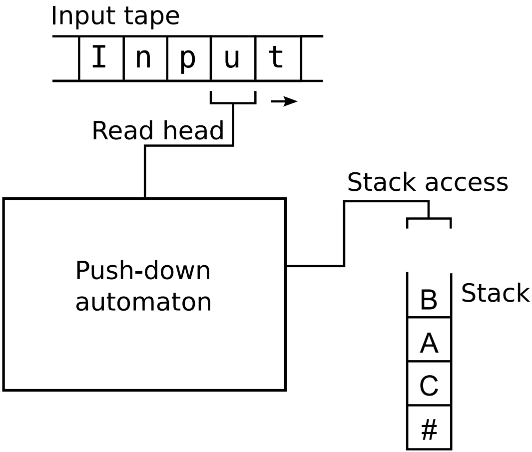
A stack is a data structure following the last-in-first-out (LIFO) principle supporting the following operations:

- push: puts an object on top of the stack
- pop: removes the object at the top of the stack



German: Keller, Stapel

Push-down Automata: Visually



German: Kellerautomat, Eingabeband, Lesekopf, Kellerzugriff

Push-down Automaton for $\{a^n b^n \mid n \in \mathbb{N}_0\}$: Idea

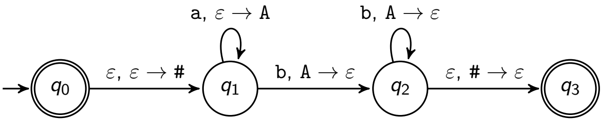
- As long as you read symbols a, push an A on the stack.
- As soon as you read a symbol b, pop an A off the stack as long as you read b.
- If reading the input is finished exactly when the stack becomes empty, accept the input.
- If there is no A to pop when reading a b, or there is still an A on the stack after reading all input symbols, or if you read an a following a b then reject the input.

Push-down Automata: Non-determinism

- PDAs are non-deterministic and can allow several next transitions from a configuration.
- Like NFAs, PDAs can have transitions that do not read a symbol from the input.
- Similarly, there can be transitions that do not pop and/or push a symbol off/to the stack.

Deterministic variants of PDAs are strictly less expressive, i. e. there are languages that can be recognized by a (non-deterministic) PDA but not the deterministic variant.

Push-down Automaton for $\{a^n b^n \mid n \in \mathbb{N}_0\}$: Diagram



Push-down Automata: Definition

Definition (Push-down Automaton)

A push-down automaton (PDA) is a 6-tuple $M = \langle Q, \Sigma, \Gamma, \delta, q_0, F\rangle$ with

- Q finite set of states
- Σ the input alphabet
- Γ the stack alphabet
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\}) \to \mathcal{P}(Q \times (\Gamma \cup \{\varepsilon\}))$ the transition function
- $q_0 \in Q$ the start state
- $F \subseteq Q$ is the set of accept states

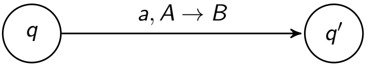
German: Kellerautomat, Eingabealphabet, Kelleralphabet, Überführungsfunktion

Push-down Automata: Transition Function

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ be a push-down automaton.

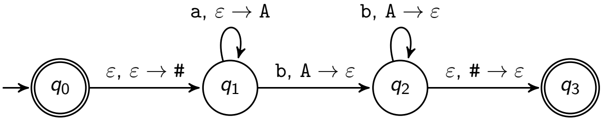
What is the Intuitive Meaning of the Transition Function δ?

- $\langle q', B \rangle \in \delta(q, a, A)$: If M is in state q, reads symbol a and has A as the topmost stack symbol, then M can transition to q' in the next step popping A off the stack and pushing B on the stack.



- special case $a = \varepsilon$ is allowed (spontaneous transition)
- special case $A = \varepsilon$ is allowed (no pop)
- special case $B = \varepsilon$ is allowed (no push)

Push-down Automaton for $\{a^n b^n \mid n \in \mathbb{N}_0\}$: Formally



$M = \langle \{q_0, q_1, q_2, q_3\}, \{a, b\}, \{A, \#\}, \delta, q_0, \{q_0, q_3\} \rangle$ with

$\delta(q_0, a, A) = \emptyset$	$\delta(q_0, b, A) = \emptyset$	$\delta(q_0, \varepsilon, A) = \emptyset$
$\delta(q_0, a, \#) = \emptyset$	$\delta(q_0, b, \#) = \emptyset$	$\delta(q_0, \varepsilon, \#) = \emptyset$
$\delta(q_0, a, \varepsilon) = \emptyset$	$\delta(q_0, b, \varepsilon) = \emptyset$	$\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \#)\}$
$\delta(q_1, a, A) = \emptyset$	$\delta(q_1, b, A) = \{(q_2, \varepsilon)\}$	$\delta(q_1, \varepsilon, A) = \emptyset$
$\delta(q_1, a, \#) = \emptyset$	$\delta(q_1, b, \#) = \emptyset$	$\delta(q_1, \varepsilon, \#) = \emptyset$
$\delta(q_1, a, \varepsilon) = \{(q_1, A)\}$	$\delta(q_1, b, \varepsilon) = \emptyset$	$\delta(q_1, \varepsilon, \varepsilon) = \emptyset$
$\delta(q_2, a, A) = \emptyset$	$\delta(q_2, b, A) = \{(q_2, \varepsilon)\}$	$\delta(q_2, \varepsilon, A) = \emptyset$
$\delta(q_2, a, \#) = \emptyset$	$\delta(q_2, b, \#) = \emptyset$	$\delta(q_2, \varepsilon, \#) = \{(q_3, \varepsilon)\}$
$\delta(q_2, a, \varepsilon) = \emptyset$	$\delta(q_2, b, \varepsilon) = \emptyset$	$\delta(q_2, \varepsilon, \varepsilon) = \emptyset$

and $\delta(q_3, x, y) = \emptyset$ for all $x \in \{a, b, \varepsilon\}$, $y \in \{A, \#, \varepsilon\}$

Push-down Automata: Accepted Words

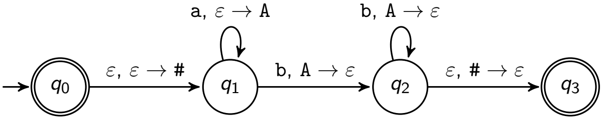
Definition

A PDA $M = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ accepts input w
if it can be written as $w = w_1 w_2 \dots w_m$ where each $w_i \in \Sigma \cup \{\varepsilon\}$
and sequences of states $r_0, r_1, \dots, r_m \in Q$ and
strings $s_0, s_1, \dots, s_m \in \Gamma^*$ exist
that satisfy the following three conditions:

1. $r_0 = q_0$ and $s_0 = \varepsilon$
2. For $i = 0, \dots, m-1$, we have $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma \cup \{\varepsilon\}$ and $t \in \Gamma^*$.
3. $r_m \in F$

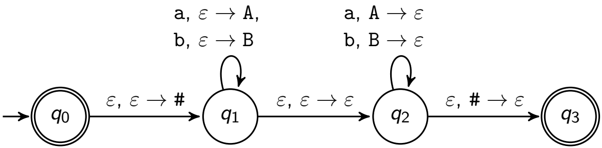
The strings s_i represent the sequence of stack contents.

Push-down Automaton for $\{a^n b^n \mid n \in \mathbb{N}_0\}$



The PDA accepts input aabb.

Acceptance: Exercise



Show that this PDA accepts input abba.

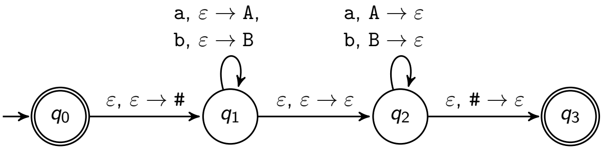
PDA: Recognized Language

Definition (Language Recognized by an NFA)

Let M be a PDA with input alphabet Σ.

The language recognized by M is defined as
$\mathcal{L}(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$.

Recognized Language: Exercise



What language does this PDA recognize?

PDAs Recognize Exactly the Context-free Languages

Theorem

A language L is context-free if and only if L is recognized by a push-down automaton.

PDAs: Exercise (if time)

Assume you want to have a possible transition from state q to state q' in your PDA that

- processes symbol c from the input word,
- can only be taken if the top stack symbol is A,
- does not pop A off the stack, and
- pushes B.

What problem do you encounter? How can you work around it?

B7.4 Summary

Summary

- Every context-free language has a grammar in Chomsky normal form. All rules have form
  - $A \rightarrow BC$ with variables A, B, C (B, C not start variable), or
  - $A \rightarrow a$ with variable A, terminal symbol a, or
  - $S \rightarrow \varepsilon$ with start variable S.
- Push-down automata (PDAs) extend NFAs with memory.
- The languages recognized by PDAs are exactly the context-free languages.