

Theory of Computer Science

A3. Proof Techniques

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A3.1 Introduction

What is a Proof?

A **mathematical proof** is

- ▶ a sequence of logical steps
- ▶ starting with one set of statements
- ▶ that comes to the conclusion
that some statement must be true.

What is a **statement**?

Mathematical Statements

Mathematical Statement

A **mathematical statement** consists of a set of **preconditions** and a set of **conclusions**.

The statement is **true** if the conclusions are true whenever the preconditions are true.

German: mathematische Aussage, Voraussetzung, Folgerung/Konklusion, wahr

Notes:

- ▶ set of preconditions is sometimes empty
- ▶ often, “assumptions” is used instead of “preconditions”; slightly unfortunate because “assumption” is also used with another meaning (\rightsquigarrow cf. indirect proofs)

Examples of Mathematical Statements

Examples (some true, some false):

- ▶ “Let $p \in \mathbb{N}_0$ be a prime number. Then p is odd.”
- ▶ “There exists an even prime number.”
- ▶ “Let $p \in \mathbb{N}_0$ with $p \geq 3$ be a prime number. Then p is odd.”
- ▶ “All prime numbers $p \geq 3$ are odd.”
- ▶ “For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ”
- ▶ “The equation $a^k + b^k = c^k$ has infinitely many solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 2$.”
- ▶ “The equation $a^k + b^k = c^k$ has no solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 3$.”

What are the preconditions, what are the conclusions?

On what Statements can we Build the Proof?

A mathematical proof is

- ▶ a sequence of logical steps
- ▶ **starting with one set of statements**
- ▶ that comes to the conclusion
that some statement must be true.

We can use:

- ▶ **axioms**: statements that are assumed to always be true
in the current context
- ▶ **theorems** and **lemmas**: statements that were already proven
 - ▶ lemma: an intermediate tool
 - ▶ theorem: itself a relevant result
- ▶ **premises**: assumptions we make
to see what consequences they have

What is a Logical Step?

A mathematical proof is

- ▶ a sequence of logical steps
- ▶ starting with one set of statements
- ▶ that comes to the conclusion that some statement must be true.

Each step **directly follows**

- ▶ from the axioms,
- ▶ premises,
- ▶ previously proven statements and
- ▶ the preconditions of the statement we want to prove.

For a formal definition, we would need formal logics.

The Role of Definitions

Definition

A **set** is an unordered collection of distinct objects.

The set that does not contain any objects is the *empty set* \emptyset .

- ▶ A definition introduces an abbreviation.
- ▶ Whenever we say “set”, we could instead say “an unordered collection of distinct objects” and vice versa.
- ▶ Definitions can also introduce notation.

Disproofs

- ▶ A **disproof** (**refutation**) shows that a given mathematical statement is **false** by giving an example where the preconditions are true, but the conclusion is false.
- ▶ This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

German: Widerlegung

- ▶ Formally, disproofs are proofs of modified (“negated”) statements.
- ▶ Be careful about how to negate a statement!

Proof Strategies

typical proof/disproof strategies:

- 1 “All $x \in S$ with the property P also have the property Q .”
“For all $x \in S$: if x has property P , then x has property Q .”
 - ▶ To prove, assume you are given an arbitrary $x \in S$ that has the property P .
Give a sequence of proof steps showing that x must have the property Q .
 - ▶ To disprove, find a **counterexample**, i. e., find an $x \in S$ that has property P but not Q and prove this.

Proof Strategies

typical proof/disproof strategies:

- ② “ A is a subset of B .”
 - ▶ To prove, assume you have an arbitrary element $x \in A$ and prove that $x \in B$.
 - ▶ To disprove, find an element in $x \in A \setminus B$ and prove that $x \in A \setminus B$.

Proof Strategies

typical proof/disproof strategies:

- ③ “For all $x \in S$: x has property P iff x has property Q .”
(“iff”: “if and only if”)
 - ▶ To prove, separately prove “if P then Q ” and “if Q then P ”.
 - ▶ To disprove, disprove “if P then Q ” or disprove “if Q then P ”.

German: “iff” = gdw. (“genau dann, wenn”)

Proof Strategies

typical proof/disproof strategies:

- ④ “ $A = B$ ”, where A and B are sets.
 - ▶ To prove, separately prove “ $A \subseteq B$ ” and “ $B \subseteq A$ ”.
 - ▶ To disprove, disprove “ $A \subseteq B$ ” or disprove “ $B \subseteq A$ ”.

Proof Techniques

most common proof techniques:

- ▶ direct proof
- ▶ indirect proof (proof by contradiction)
- ▶ proof by contrapositive
- ▶ mathematical induction
- ▶ structural induction

German: direkter Beweis, indirekter Beweis (Beweis durch Widerspruch), Kontraposition, vollständige Induktion, strukturelle Induktion

Exercise

Negate the following statement:

If the sun is shining then all kids eat ice cream.



A3.2 Direct Proof

Direct Proof

Direct Proof

Direct derivation of the statement by deducing or rewriting.

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

We first show that $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$ (\subseteq part):

Let $x \in A \cap (B \cup C)$. Then by the definition of \cap it holds that $x \in A$ and $x \in B \cup C$.

We make a case distinction between $x \in B$ and $x \notin B$:

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

Otherwise, because $x \in B \cup C$ we know that $x \in C$ and thus with $x \in A$, that $x \in A \cap C$.

In both cases $x \in A \cap B$ or $x \in A \cap C$,
and we conclude $x \in (A \cap B) \cup (A \cap C)$.

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Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

\supseteq part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.

The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

If $x \notin A \cap B$ we know $x \in A \cap C$ due to $x \in (A \cap B) \cup (A \cap C)$.

This (analogously) implies $x \in A$ and $x \in C$, and hence $x \in B \cup C$ and thus $x \in A \cap (B \cup C)$.

In both cases we conclude $x \in A \cap (B \cup C)$.

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Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

We have shown that every element of $A \cap (B \cup C)$ is an element of $(A \cap B) \cup (A \cap C)$ and vice versa.

Thus, both sets are equal. □

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

Alternative:

$$\begin{aligned} A \cap (B \cup C) &= \{x \mid x \in A \text{ and } x \in B \cup C\} \\ &= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\} \\ &= \{x \mid (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\} \\ &= \{x \mid x \in A \cap B \text{ or } x \in A \cap C\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$



A3.3 Indirect Proof

Indirect Proof

Indirect Proof (Proof by Contradiction)

- ▶ Make an **assumption** that the statement is false.
- ▶ Derive a **contradiction** from the assumption together with the preconditions of the statement.
- ▶ This shows that the assumption must be false given the preconditions of the statement, and hence the original statement must be true.

German: Annahme, Widerspruch

Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

Proof.

Assumption: There are only finitely many prime numbers.

Let $P = \{p_1, \dots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \dots \cdot p_n + 1$.

Since $m \geq 2$, it must have a prime factor.

Let p be such a prime factor.

Since p is a prime number, p has to be in P .

The number m is not divisible without remainder by any of the numbers in P . Hence p is no factor of m .

\rightsquigarrow **Contradiction**



A3.4 Contrapositive

Proof by Contrapositive

Proof by Contrapositive

Prove “If A , then B ” by proving “If not B , then not A .”

German: (Beweis durch) Kontraposition

Examples:

- ▶ Prove “For all $n \in \mathbb{N}_0$: if n^2 is odd, then n is odd” by proving “For all $n \in \mathbb{N}_0$, if n is even, then n^2 is even.”
- ▶ Prove “For all $n \in \mathbb{N}_0$: if n is not a square number, then \sqrt{n} is irrational” by proving “For all $n \in \mathbb{N}_0$: if \sqrt{n} is rational, then n is a square number.”

Exercise

How would you prove the following statement by contrapositive:

If the sun is shining then all kids eat ice cream.



A3.5 Mathematical Induction

Mathematical Induction

Mathematical Induction

Proof of a statement for all natural numbers n with $n \geq m$

- ▶ **basis**: proof of the statement for $n = m$
- ▶ **induction hypothesis (IH)**:
suppose that the statement is true for all k with $m \leq k \leq n$
- ▶ **inductive step**: proof of the statement for $n + 1$
using the induction hypothesis

German: vollständige Induktion, Induktionsanfang,
Induktionsvoraussetzung, Induktionsschritt

Mathematical Induction: Example

Theorem

For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^n (2k - 1) = n^2$

Proof.

Mathematical induction over n :

basis $n = 1$: $\sum_{k=1}^1 (2k - 1) = 2 - 1 = 1 = 1^2$

IH: $\sum_{k=1}^m (2k - 1) = m^2$ for all $1 \leq m \leq n$

inductive step $n \rightarrow n + 1$:

$$\begin{aligned}\sum_{k=1}^{n+1} (2k - 1) &= \left(\sum_{k=1}^n (2k - 1) \right) + 2(n + 1) - 1 \\ &\stackrel{\text{IH}}{=} n^2 + 2(n + 1) - 1 \\ &= n^2 + 2n + 1 = (n + 1)^2\end{aligned}$$



A3.6 Structural Induction

Inductively Defined Sets: Examples

Example (Natural Numbers)

The set \mathbb{N}_0 of natural numbers is inductively defined as follows:

- ▶ 0 is a natural number.
- ▶ If n is a natural number, then $n + 1$ is a natural number.

Example (Binary Tree)

The set \mathcal{B} of binary trees is inductively defined as follows:

- ▶ \square is a binary tree (a **leaf**)
- ▶ If L and R are binary trees, then $\langle L, \bigcirc, R \rangle$ is a binary tree (with **inner node** \bigcirc).

German: Binärbaum, Blatt, innerer Knoten

Implicit statement: all elements of the set can be constructed by finite application of these rules

Inductive Definition of a Set

Inductive Definition

A set M can be defined **inductively** by specifying

- ▶ **basic elements** that are contained in M
- ▶ **construction rules** of the form
“Given some elements of M , another element of M can be constructed like this.”

German: induktive Definition, Basiselemente, Konstruktionsregeln

Structural Induction

Structural Induction

Proof of statement for all elements of an inductively defined set

- ▶ **basis**: proof of the statement for the basic elements
- ▶ **induction hypothesis (IH)**:
suppose that the statement is true for some elements M
- ▶ **inductive step**: proof of the statement for elements constructed by applying a construction rule to M
(one inductive step for each construction rule)

German: strukturelle Induktion, Induktionsanfang, Induktionsvoraussetzung, Induktionsschritt

Structural Induction: Example (1)

Definition (Leaves of a Binary Tree)

The number of **leaves** of a binary tree B , written $leaves(B)$, is defined as follows:

$$leaves(\square) = 1$$

$$leaves(\langle L, \circ, R \rangle) = leaves(L) + leaves(R)$$

Definition (Inner Nodes of a Binary Tree)

The number of **inner nodes** of a binary tree B , written $inner(B)$, is defined as follows:

$$inner(\square) = 0$$

$$inner(\langle L, \circ, R \rangle) = inner(L) + inner(R) + 1$$

Structural Induction: Example (2)

Theorem

For all binary trees B : $inner(B) = leaves(B) - 1$.

Proof.

induction basis:

$$inner(\square) = 0 = 1 - 1 = leaves(\square) - 1$$

\rightsquigarrow statement is true for base case

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Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree $\langle L, \circlearrowleft, R \rangle$, we may use that it is true for the subtrees L and R .

inductive step for $B = \langle L, \circlearrowleft, R \rangle$:

$$\begin{aligned} \mathit{inner}(B) &= \mathit{inner}(L) + \mathit{inner}(R) + 1 \\ &\stackrel{\text{IH}}{=} (\mathit{leaves}(L) - 1) + (\mathit{leaves}(R) - 1) + 1 \\ &= \mathit{leaves}(L) + \mathit{leaves}(R) - 1 = \mathit{leaves}(B) - 1 \end{aligned}$$



Structural Induction: Exercise (if time)

Definition (Height of a Binary Tree)

The **height** of a binary tree B , written $height(B)$, is defined as follows:

$$height(\square) = 0$$
$$height(\langle L, \circ, R \rangle) = \max\{height(L), height(R)\} + 1$$

Prove by structural induction:

Theorem

For all binary trees B : $leaves(B) \leq 2^{height(B)}$.



A3.7 Summary

Summary

- ▶ A **proof** is based on axioms and previously proven statements.
- ▶ Individual **proof steps** must be obvious derivations.
- ▶ **direct proof**: sequence of derivations or rewriting
- ▶ **indirect proof**: refute the negated statement
- ▶ **contrapositive**: prove " $A \Rightarrow B$ " as "not $B \Rightarrow$ not A "
- ▶ **mathematical induction**: prove statement for a starting point and show that it always carries over to the next number
- ▶ **structural induction**: generalization of mathematical induction to arbitrary recursive structures