Theory of Computer Science

A2. Mathematical Foundations

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Sets, Tuples, Relations

Sets

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  - explicit, listing all elements, e. g. $A = \{1, 2, 3\}$
  - implicit, specifying a property characterizing all elements, e. g. $A = \{x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 3\}$
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German: Menge, Element, leere Menge, Mächtigkeit/Kardinalität

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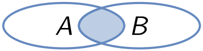
- Cardinality of power set of finite set S: $|\mathcal{P}(S)| =$

German: Teilmenge, echte Teilmenge, Potenzmenge

Set Operations

- intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



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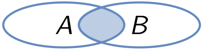

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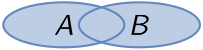

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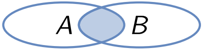

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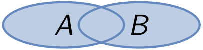

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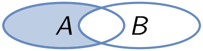

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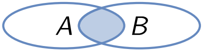

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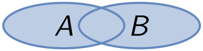

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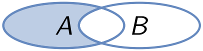

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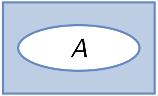

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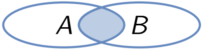

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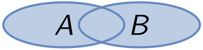

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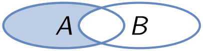

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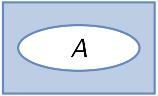

German: Schnitt, Vereinigung, Differenz, Komplement

Tuples

- *k-tuple*: ordered sequence of k objects
- written $(o_1, \dots, o_k)$ or $\langle o_1, \dots, o_k \rangle$
- unlike sets, order matters ($\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$)
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- objects contained in tuples are called components
- terminology:
  - $k = 2$: (ordered) pair
  - $k = 3$: triple
  - more rarely: quadruple, quintuple, sextuple, septuple, ...
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German: k-Tupel, Komponente, Paar, Tripel

Cartesian Product

- for sets $M_1, M_2, \dots, M_n$, the Cartesian product $M_1 \times \dots \times M_n$ is the set $M_1 \times \dots \times M_n = \{\langle o_1, \dots, o_n \rangle \mid o_1 \in M_1, \dots, o_n \in M_n\}$.
- Example: $M_1 = \{a, b, c\}, M_2 = \{1, 2\}$, $M_1 \times M_2 = \{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \langle c, 1\rangle, \langle c, 2\rangle\}$

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- special case: $M^k = M \times \cdots \times M$ (k times)
- example with $M = \{1, 2\}$: $M^2 = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$

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German: kartesisches Produkt

Relations

- an n-ary relation R over the sets $M_1, \ldots, M_n$ is a subset of their Cartesian product: $R \subseteq M_1 \times \cdots \times M_n$.
- example with $M = \{1, 2\}$:
  $R_{\leq} \subseteq M^2$ as $R_{\leq} = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$

German: (n-stellige) Relation

Exercise

Consider $S = \mathcal{P}(\{1, 2\}) \times \{a, b\}$.

1. Write down three different elements of S.
2. What is $|S|$?

Functions

Functions

Definition (Total Function)

A (total) function $f : D \to C$ (with sets D, C)
maps every value of its domain D
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- $add : \mathbb{N}_0^2 \to \mathbb{N}_0$ with $add(x, y) = x + y$
- $add_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}$ with $add_{\mathbb{R}}(x, y) = x + y$

Functions: Example

Example

Let $Q = \{q_0, q_1, q_2, q_{\text{accept}}, q_{\text{reject}}\}$ and $\Gamma = \{0, 1, \square\}$.

Define $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ by

δ	0	1	$\square$
q_0	$\langle q_0, 0, \mathrm{R}\rangle$	$\langle q_0, 1, \mathrm{R}\rangle$	$\langle q_1, \square, \mathrm{L}\rangle$
q_1	$\langle q_2, 1, \mathrm{L}\rangle$	$\langle q_1, 0, \mathrm{L}\rangle$	$\langle q_{\text{reject}}, 1, \mathrm{L}\rangle$
q_2	$\langle q_2, 0, \mathrm{L}\rangle$	$\langle q_2, 1, \mathrm{L}\rangle$	$\langle q_{\text{accept}}, \square, \mathrm{R}\rangle$

Then, e. g., $\delta(q_0, 1) = \langle q_0, 1, \mathrm{R}\rangle$

Partial Functions

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If f does not map $x \in X$ to any value in Y, then f is undefined for x.

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Example

$f : \mathbb{N}_0 \times \mathbb{N}_0 \to_p \mathbb{N}_0$ with

$$f(x, y) = \begin{cases} x - y & \text{if } y \leq x \\ \text{undefined} & \text{otherwise} \end{cases}$$

Exercises

Let $V = \{X, Y, Z\}$, $\Sigma = \{a, b, c\}$ and $Q = \{q_1, q_2\}$ be three sets.

1. Specify a non-trivial example for a partial function $\delta : Q \times \Sigma \to_{\mathrm{p}} \mathcal{P}(Q)$.

2. Specify a non-trivial example for a relation $P \subseteq (V \cup \Sigma)^2 \times V^2$.

Summary

Summary

- sets: unordered, contain every element at most once
- tuples: ordered, can contain the same object multiple times
- Cartesian product: $M_1 \times \cdots \times M_n$ set of all n-tuples where the i-th component is in M_i
- function $f : X \to Y$ maps every value in X to exactly one value in Y
- partial function $g : X \to_{\mathrm{p}} Y$ may be undefined for some values in X