

Theory of Computer Science

E4. Some NP-Complete Problems, Part I

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May 18, 2020

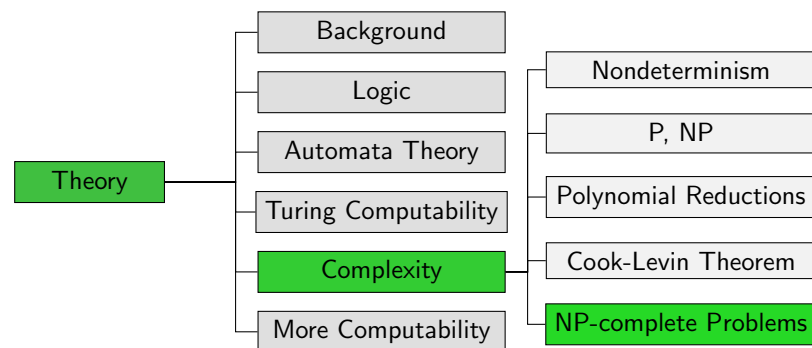
Theory of Computer Science

May 18, 2020 — E4. Some NP-Complete Problems, Part I

E4.1 Graph Problems

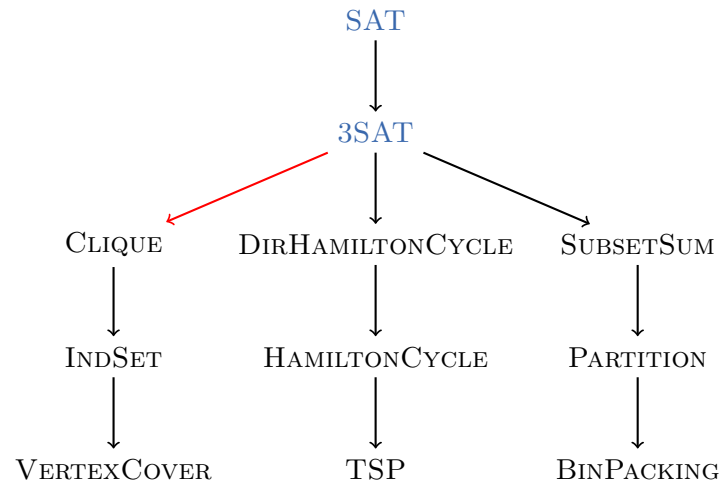
E4.2 Routing Problems

Course Overview



E4.1 Graph Problems

3SAT \leq_p CLIQUE



CLIQUE

Definition (CLIQUE)

The problem **CLIQUE** is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$, number $K \in \mathbb{N}_0$

Question: Does G have a clique of size at least K , i. e., a set of vertices $C \subseteq V$ with $|C| \geq K$ and $\{u, v\} \in E$ for all $u, v \in C$ with $u \neq v$?

German: Clique

CLIQUE is NP-Complete (1)

Theorem (CLIQUE is NP-Complete)

CLIQUE is NP-complete.

CLIQUE is NP-Complete (2)

Proof.

CLIQUE \in NP: guess and check.

CLIQUE is NP-hard: We show $3SAT \leq_p CLIQUE$.

- ▶ We are given a 3-CNF formula φ , and we may assume that each clause has exactly three literals.
 - ▶ In polynomial time, we must construct a graph $G = \langle V, E \rangle$ and a number K such that: G has a clique of size at least K iff φ is satisfiable.
- ↪ construction of V, E, K on the following slides.

...

CLIQUE is NP-Complete (3)

Proof (continued).

Let m be the number of clauses in φ .

Let l_{ij} the j -th literal in clause i .

Define V , E , K as follows:

- ▶ $V = \{\langle i, j \rangle \mid 1 \leq i \leq m, 1 \leq j \leq 3\}$
 \rightsquigarrow a vertex for every literal of every clause
- ▶ E contains edge between $\langle i, j \rangle$ and $\langle i', j' \rangle$ if and only if
 - ▶ $i \neq i' \rightsquigarrow$ belong to **different clauses**, and
 - ▶ l_{ij} and $l_{i'j'}$ are **not complementary literals**
- ▶ $K = m$

\rightsquigarrow obviously polynomially computable

to show: reduction property

...

CLIQUE is NP-Complete (4)

Proof (continued).

(\Rightarrow): If φ is satisfiable, then $\langle V, E \rangle$ has clique of size at least K :

- ▶ Given a satisfying variable assignment choose a vertex corresponding to a satisfied literal in each clause.
- ▶ The chosen K vertices are all connected with each other and hence form a clique of size K .

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CLIQUE is NP-Complete (5)

Proof (continued).

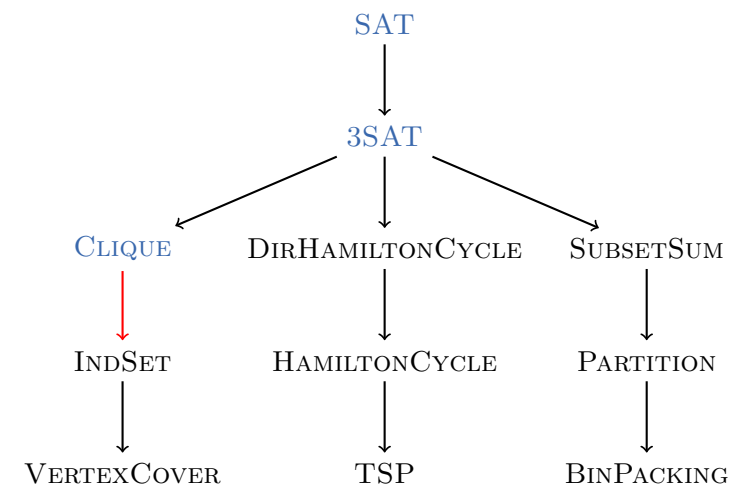
(\Leftarrow): If $\langle V, E \rangle$ has a clique of size at least K , then φ is satisfiable:

- ▶ Consider a given clique C of size at least K .
 - ▶ The vertices in C must all correspond to different clauses (vertices in the same clause are not connected by edges).
- \rightsquigarrow exactly one vertex per clause is included in C
- ▶ Two vertices in C never correspond to complementary literals X and $\neg X$ (due to the way we defined the edges).
 - ▶ If a vertex corresp. to X was chosen, map X to 1 (true).
 - ▶ If a vertex corresp. to $\neg X$ was chosen, map X to 0 (false).
 - ▶ If neither was chosen, arbitrarily map X to 0 or 1.

\rightsquigarrow satisfying assignment

□

CLIQUE \leq_p INDSET



INDSET

Definition (INDSET)

The problem **INDSET** is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$, number $K \in \mathbb{N}_0$

Question: Does G have an independent set of size at least K , i. e., a set of vertices $I \subseteq V$ with $|I| \geq K$ and $\{u, v\} \notin E$ for all $u, v \in I$ with $u \neq v$?

German: unabhängige Menge

INDSET is NP-Complete (1)

Theorem (INDSET is NP-Complete)

INDSET is NP-complete.

INDSET is NP-Complete (2)

Proof.

INDSET \in NP: guess and check.

INDSET is NP-hard: We show $\text{CLIQUE} \leq_p \text{INDSET}$.

We describe a polynomial reduction f .

Let $\langle G, K \rangle$ with $G = \langle V, E \rangle$ be the given input for **CLIQUE**.

Then $f(\langle G, K \rangle)$ is the **INDSET** instance $\langle \bar{G}, K \rangle$, where $\bar{G} := \langle V, \bar{E} \rangle$ and $\bar{E} := \{\{u, v\} \subseteq V \mid u \neq v, \{u, v\} \notin E\}$.
(This graph \bar{G} is called the **complement graph** of G .)

Clearly f can be computed in polynomial time. ...

INDSET is NP-Complete (3)

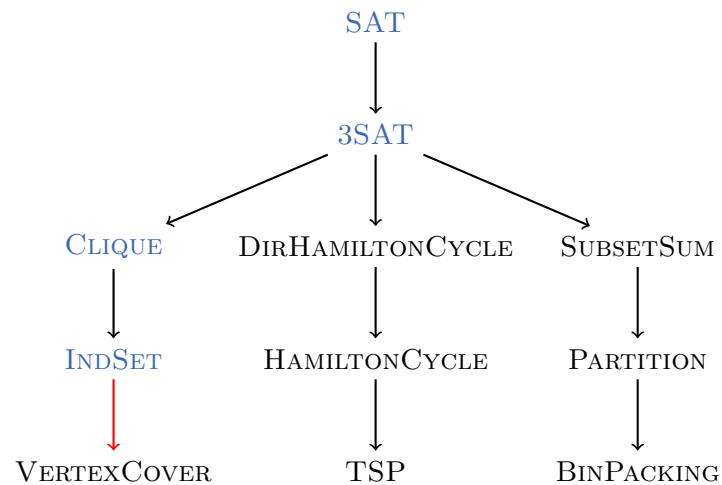
Proof (continued).

We have:

- $\langle \langle V, E \rangle, K \rangle \in \text{CLIQUE}$
- iff there exists a set $V' \subseteq V$ with $|V'| \geq K$
and $\{u, v\} \in E$ for all $u, v \in V'$ with $u \neq v$
- iff there exists a set $V' \subseteq V$ with $|V'| \geq K$
and $\{u, v\} \notin \bar{E}$ for all $u, v \in V'$ with $u \neq v$
- iff $\langle \langle V, \bar{E} \rangle, K \rangle \in \text{INDSET}$
- iff $f(\langle \langle V, E \rangle, K \rangle) \in \text{INDSET}$

and hence f is a reduction. \square

INDSET \leq_p VERTEXCOVER



VERTEXCOVER

Definition (VERTEXCOVER)

The problem **VERTEXCOVER** is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$, number $K \in \mathbb{N}_0$

Question: Does G have a vertex cover of size at most K , i. e., a set of vertices $C \subseteq V$ with $|C| \leq K$ and $\{u, v\} \cap C \neq \emptyset$ for all $\{u, v\} \in E$?

German: Knotenüberdeckung

VERTEXCOVER is NP-Complete (1)

Theorem (VERTEXCOVER is NP-Complete)

VERTEXCOVER is NP-complete.

VERTEXCOVER is NP-Complete (2)

Proof.

VERTEXCOVER \in NP: guess and check.

VERTEXCOVER is NP-hard:

We show **INDSET** \leq_p **VERTEXCOVER**.

We describe a polynomial reduction f .

Let $\langle G, K \rangle$ with $G = \langle V, E \rangle$ be the given input for **INDSET**.

Then $f(\langle G, K \rangle) := \langle G, |V| - K \rangle$.

This can clearly be computed in polynomial time. ...

VERTEXCOVER is NP-Complete (3)

Proof (continued).

For vertex set $V' \subseteq V$, we write $\overline{V'}$ for its **complement** $V \setminus V'$.

Observation: a set of vertices is a vertex cover
iff its complement is an independent set.

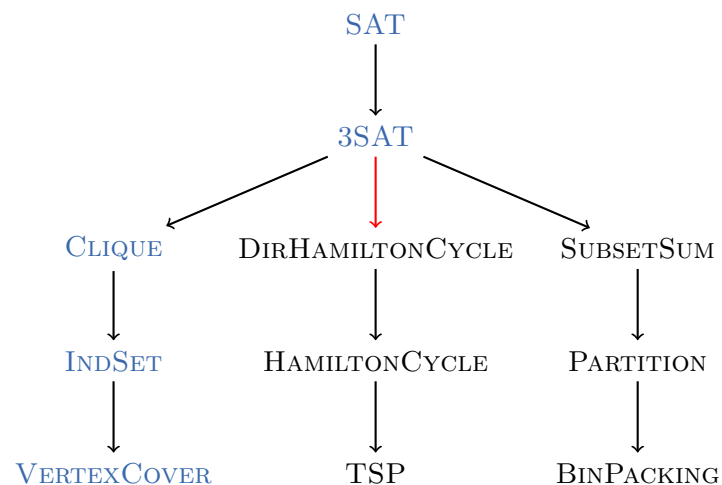
We thus have:

- $\langle \langle V, E \rangle, K \rangle \in \text{INDSET}$
- iff $\langle V, E \rangle$ has an independent set I with $|I| \geq K$
- iff $\langle V, E \rangle$ has a vertex cover C with $|\overline{C}| \geq K$
- iff $\langle V, E \rangle$ has a vertex cover C with $|C| \leq |V| - K$
- iff $\langle \langle V, E \rangle, |V| - K \rangle \in \text{VERTEXCOVER}$
- iff $f(\langle \langle V, E \rangle, K \rangle) \in \text{VERTEXCOVER}$

and hence f is a reduction. □

E4.2 Routing Problems

$3\text{SAT} \leq_p \text{DIRHAMILTONCYCLE}$



DIRHAMILTONCYCLE is NP-Complete (1)

Definition (Reminder: DIRHAMILTONCYCLE)

The problem **DIRHAMILTONCYCLE** is defined as follows:

Given: directed graph $G = \langle V, E \rangle$

Question: Does G contain a Hamilton cycle?

Theorem

DIRHAMILTONCYCLE is NP-complete.

DIRHAMILTONCYCLE is NP-Complete (2)

Proof.

DIRHAMILTONCYCLE \in NP: guess and check.

DIRHAMILTONCYCLE is NP-hard:

We show $3SAT \leq_p$ DIRHAMILTONCYCLE.

- ▶ We are given a 3-CNF formula φ where each clause contains exactly three literals and no clause contains duplicated literals.
- ▶ We must, in polynomial time, construct a directed graph $G = \langle V, E \rangle$ such that: G contains a Hamilton cycle iff φ is satisfiable.
- ▶ construction of $\langle V, E \rangle$ on the following slides

...

DIRHAMILTONCYCLE is NP-Complete (3)

Proof (continued).

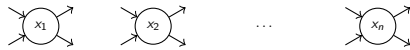
- ▶ Let X_1, \dots, X_n be the propositional variables in φ .
- ▶ Let c_1, \dots, c_m be the clauses of φ with $c_i = (l_{i1} \vee l_{i2} \vee l_{i3})$.
- ▶ Construct a graph with $6m + n$ vertices (described on the following slides).

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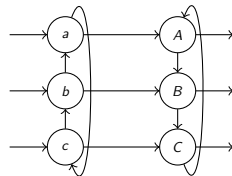
DIRHAMILTONCYCLE is NP-Complete (4)

Proof (continued).

- ▶ For every variable X_i , add vertex x_i with 2 incoming and 2 outgoing edges:



- ▶ For every clause c_j , add the subgraph C_j with 6 vertices:



- ▶ We describe later how to connect these parts.

...

DIRHAMILTONCYCLE is NP-Complete (5)

Proof (continued).

Let π be a Hamilton cycle of the total graph.

- ▶ Whenever π enters subgraph C_j from one of its "entrances", it must leave via the corresponding "exit": $(a \rightarrow A, b \rightarrow B, c \rightarrow C)$. Otherwise, π cannot be a Hamilton cycle.
- ▶ Hamilton cycles can behave in the following ways with regard to C_j :
 - ▶ π passes through C_j once (from any entrance)
 - ▶ π passes through C_j twice (from any two entrances)
 - ▶ π passes through C_j three times (once from every entrance)

...

DIRHAMILTONCYCLE is NP-Complete (6)

Proof (continued).

Connect the “open ends” in the graph as follows:

- ▶ Identify entrances/exits of the clause subgraph C_j with the three literals in clause c_j .
- ▶ One exit of x_i is **positive**, the other one is **negative**.
- ▶ For the **positive** exit, determine the clauses in which the positive literal X_i occurs:
 - ▶ Connect the positive exit of x_i with the X_i -entrance of the first such clause graph.
 - ▶ Connect the X_i -exit of this clause graph with the X_i -entrance of the second such clause graph, and so on.
 - ▶ Connect the X_i -exit of the last such clause graph with the positive entrance of x_{i+1} (or x_1 if $i = n$).
- ▶ analogously for the **negative** exit of x_i and the literal $\neg X_i$

...

DIRHAMILTONCYCLE is NP-Complete (7)

Proof (continued).

The construction is polynomial and is a reduction:

 (\Rightarrow) : **construct a Hamilton cycle from a satisfying assignment**

- ▶ Given a satisfying assignment \mathcal{I} , construct a Hamilton cycle that leaves x_i through the positive exit if $\mathcal{I}(X_i)$ is true and by the negative exit if $\mathcal{I}(X_i)$ is false.
- ▶ Afterwards, we visit all C_j -subgraphs for clauses that are satisfied by this literal.
- ▶ In total, we visit each C_j -subgraph 1–3 times.

...

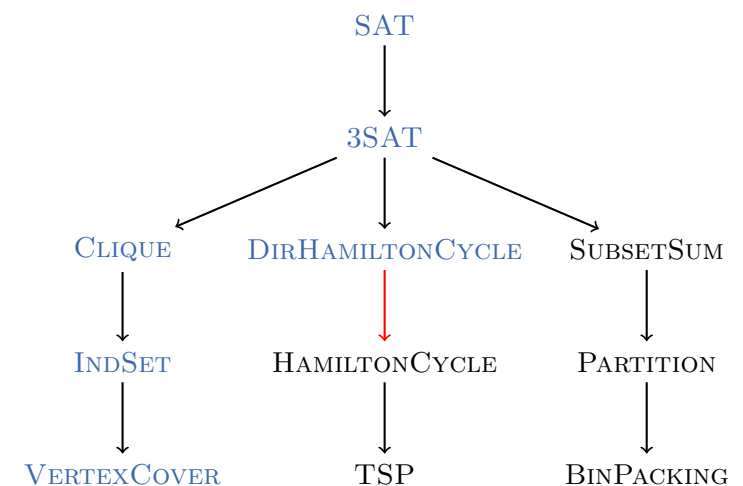
DIRHAMILTONCYCLE is NP-Complete (8)

Proof (continued).

 (\Leftarrow) : **construct a satisfying assignment from a Hamilton cycle**

- ▶ A Hamilton cycle visits every vertex x_i and leaves it by the positive or negative exit.
- ▶ Map X_i to true or false depending on which exit is used to leave x_i .
- ▶ Because the cycle must traverse each C_j -subgraph at least once (otherwise it is not a Hamilton cycle), this results in a satisfying assignment. (Details omitted.)

□

DIRHAMILTONCYCLE \leq_p HAMILTONCYCLE

HAMILTONCYCLE is NP-Complete (1)

Definition (Reminder: HAMILTONCYCLE)

The problem **HAMILTONCYCLE** is defined as follows:

Given: undirected graph $G = \langle V, E \rangle$

Question: Does G contain a Hamilton cycle?

Theorem

HAMILTONCYCLE is NP-complete.

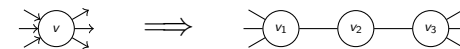
HAMILTONCYCLE is NP-Complete (2)

Proof sketch.

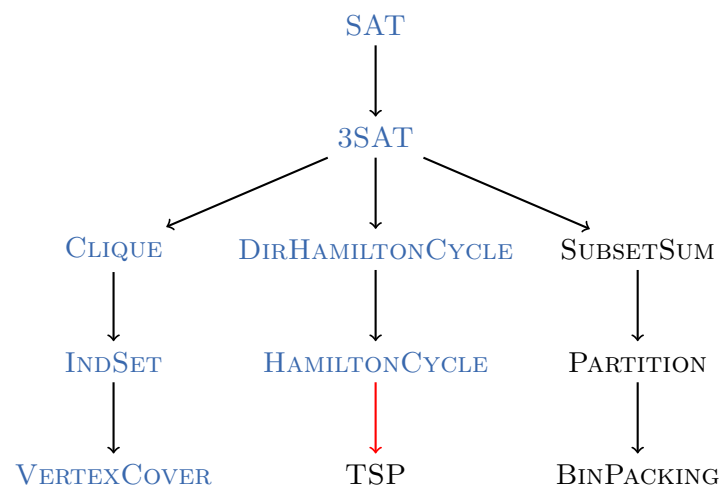
HAMILTONCYCLE \in NP: guess and check.

HAMILTONCYCLE is NP-hard: We show
 $\text{DIRHAMILTONCYCLE} \leq_p \text{HAMILTONCYCLE}$.

Basic building block of the reduction:



HAMILTONCYCLE \leq_p TSP



TSP is NP-Complete (1)

Definition (Reminder: TSP)

TSP (traveling salesperson problem) is the following decision problem:

- ▶ **Given:** finite set $S \neq \emptyset$ of cities, symmetric cost function $\text{cost} : S \times S \rightarrow \mathbb{N}_0$, cost bound $K \in \mathbb{N}_0$
- ▶ **Question:** Is there a tour with total cost at most K , i. e., a permutation $\langle s_1, \dots, s_n \rangle$ of the cities with $\sum_{i=1}^{n-1} \text{cost}(s_i, s_{i+1}) + \text{cost}(s_n, s_1) \leq K$?

German: Problem der/des Handlungsreisenden

Theorem

TSP is NP-complete.

TSP is NP-Complete (2)

Proof.

TSP \in NP: guess and check.

TSP is NP-hard: We showed $\text{HAMILTONCYCLE} \leq_p \text{TSP}$
in Chapter E2. □

Summary

- ▶ In this chapter we showed NP-completeness of
 - ▶ three classical graph problems:
CLIQUE, INDSET, VERTEXCOVER
 - ▶ three classical routing problems:
DIRHAMILTONCYCLE, HAMILTONCYCLE, TSP