Theory of Computer Science

E1. Complexity Theory: Motivation and Introduction

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Overview: Course

contents of this course:

A. background ✓
   - ▷ mathematical foundations and proof techniques

B. logic ✓
   - ▷ How can knowledge be represented? How can reasoning be automated?

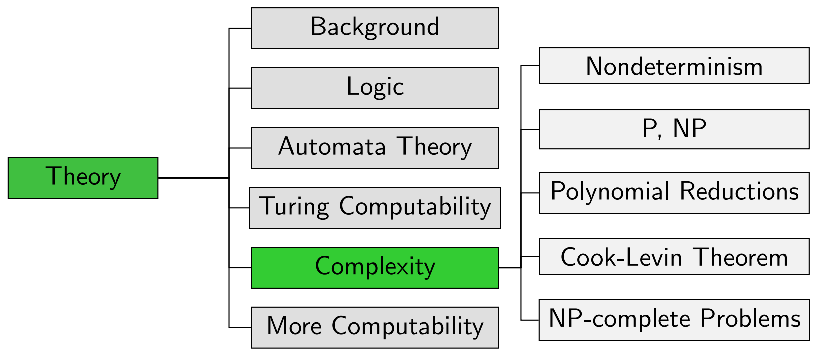
C. automata theory and formal languages ✓
   - ▷ What is a computation?

D. Turing computability ✓
   - ▷ What can be computed at all?

E. complexity theory
   - ▷ What can be computed efficiently?

F. more computability theory
   - ▷ Other models of computability

Course Overview



E1.1 Motivation

A Scenario (1)

Example Scenario

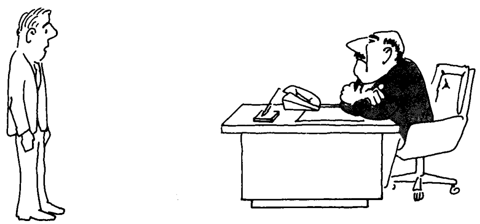
- You are a programmer at a logistics company.
- Your boss gives you the task of developing a program to optimize the route of a delivery truck:
  - The truck begins its route at the company depot.
  - It has to visit 50 stops.
  - You know the distances between all relevant locations (stops and depot).
  - Your program should compute a tour visiting all stops and returning to the depot on a shortest route.

A Scenario (2)

Example Scenario (ctd.)

- You work on the problem for weeks, but you do not manage to complete the task.
- All of your attempted programs
  - compute routes that are possibly suboptimal, or
  - do not terminate in reasonable time (say: within a month).
- What do you say to your boss?

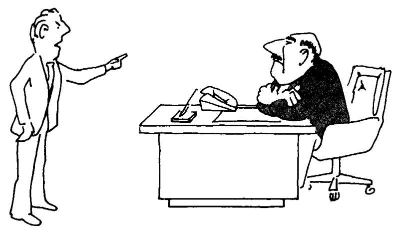
What You Don't Want to Say



"I can't find an efficient algorithm,
I guess I'm just too dumb."

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 2

What You Would Like to Say



"I can't find an efficient algorithm,
because no such algorithm is possible!"

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 2

What Complexity Theory Allows You to Say

"I can't find an efficient algorithm,
but neither can all these famous people."

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 3

Why Complexity Theory?

Complexity Theory

Complexity theory tells us which problems can be solved quickly ("simple problems") and which ones cannot ("hard problems").

German: Komplexitätstheorie

- This is useful in practice because simple and hard problems require different techniques to solve.
- If we can show that a problem is hard we do not need to waste our time with the (futile) search for a "simple" algorithm.

Why Reductions?

Reductions

An important part of complexity theory are (polynomial) reductions that show how a given problem P can be reduced to another problem Q.

German: Reduktionen

- useful for theoretical analysis of P and Q because it allows us to transfer our knowledge between them
- often also useful for practical algorithms for P: reduce P to Q and then use the best known algorithm for Q

Test Your Intuition! (1)

- The following slide lists some graph problems.
- The input is always a directed graph $G = \langle V, E \rangle$.
- How difficult are the problems in your opinion?
- Sort the problems
  from easiest (= requires least amount of time to solve)
  to hardest (= requires most time to solve)
- no justification necessary, just follow your intuition!
- anonymous and not graded

Test Your Intuition! (2)

1. Find a simple path (= without cycle) from $u \in V$ to $v \in V$ with minimal length.
2. Find a simple path (= without cycle) from $u \in V$ to $v \in V$ with maximal length.
3. Determine whether G is strongly connected (every node is reachable from every other node).
4. Find a cycle (non-empty path from u to u for any $u \in V$; multiple visits of nodes are allowed).
5. Find a cycle that visits all nodes.
6. Find a cycle that visits a given node u.
7. Find a path that visits all nodes without repeating a node.
8. Find a path that uses all edges without repeating an edge.

E1.2 How to Measure Runtime?

How to Measure Runtime?

- Time complexity is a way to measure how much time it takes to solve a problem.
- How can we define such a measure appropriately?

German: Zeitkomplexität/Zeitaufwand

Example Statements about Runtime

Example statements about runtime:

- "Running `sort /usr/share/dict/words` on the computer `dakar` takes 0.035 seconds."
- "With a 1 MiB input file, `sort` takes at most 1 second on a modern computer."
- "Quicksort is faster than sorting by insertion."
- "Sorting by insertion is slow."

⇝ Very different statements with different pros and cons.

Precise Statements vs. General Statements

Example Statement about Runtime
"Running `sort /usr/share/dict/words` on the computer `dakar` takes 0.035 seconds."

advantage: very precise

disadvantage: not general

- input-specific:
  What if we want to sort other files?
- machine-specific:
  What happens on a different computer?
- even situation-specific:
  Will we get the same result tomorrow that we got today?

General Statements about Runtime

In this course we want to make general statements about runtime. We accomplish this in three ways:

1. General Inputs
Instead of concrete inputs, we talk about general types of input:

- Example: runtime to sort an input of size n in the worst case
- Example: runtime to sort an input of size n in the average case

here: runtime for input size n in the worst case

General Statements about Runtime

In this course we want to make general statements about runtime. We accomplish this in three ways:

2. Ignoring Details
Instead of exact formulas for the runtime we specify the order of magnitude:

- Example: instead of saying that we need time $\lceil 1.2n \log n \rceil - 4n + 100$, we say that we need time $O(n \log n)$.
- Example: instead of saying that we need time $O(n \log n)$, $O(n^2)$ or $O(n^4)$, we say that we need polynomial time.

here: What can be computed in polynomial time?

General Statements about Runtime

In this course we want to make general statements about runtime. We accomplish this in three ways:

3. Abstract Cost Measures

Instead of the runtime on a concrete computer we consider a more abstract cost measure:

- Example: count the number of executed machine code statements
- Example: count the number of executed Java byte code statements
- Example: count the number of element comparisons of a sorting algorithms

here: count the computation steps of a Turing machine (polynomially equivalent to other measures)

E1.3 Decision Problems

Decision Problems

- As before, we simplify our investigation by restricting our attention to decision problems.
- More complex computational problems can be solved with multiple queries for an appropriately defined decision problem ("playing 20 questions").
- Formally, decision problems are languages (as before), but we use an informal "given" / "question" notation where possible.

Example: Decision vs. General Problem (1)

Definition (Hamilton Cycle)

Let $G = \langle V, E \rangle$ be a (directed or undirected) graph.

A Hamilton cycle of G is a sequence of vertices in V, $\pi = \langle v_0, \dots, v_n \rangle$, with the following properties:

- π is a path: there is an edge from v_i to v_{i+1} for all $0 \leq i < n$
- π is a cycle: $v_0 = v_n$
- π is simple: $v_i \neq v_j$ for all $i \neq j$ with $i, j < n$
- π is Hamiltonian: all nodes of V are included in π

German: Hamiltonkreis/Hamiltonzyklus

Example: Decision vs. General Problem (2)

Example (Hamilton Cycles in Directed Graphs)

$\mathcal{P}$: general problem DirHamiltonCycleGen
- Input: directed graph $G = \langle V, E \rangle$
- Output: a Hamilton cycle of G or a message that none exists

$\mathcal{D}$: decision problem DirHamiltonCycle
- Given: directed graph $G = \langle V, E \rangle$
- Question: Does G contain a Hamilton cycle?

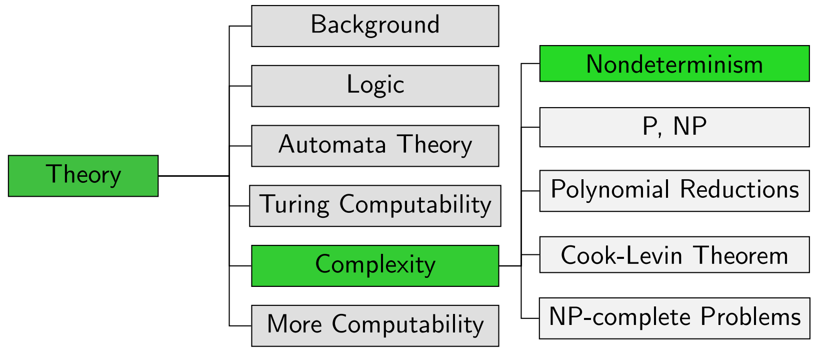
These problems are polynomially equivalent:
from a polynomial algorithm for one of the problems
one can construct a polynomial algorithm for the other problem.
(Without proof.)

Algorithms for Decision Problems

Algorithms for decision problems:
- Where possible, we specify algorithms for decision problems in pseudo-code.
- Since they are only yes/no questions, we do not have to return a general result.
- Instead we use the statements
  - **ACCEPT** to accept the given input ("yes" answer) and
  - **REJECT** to reject it ("no" answer).
- Where we must be more formal, we use Turing machines and the notion of accepting from chapter C7.

E1.4 Nondeterminism

Course Overview



Nondeterminism

- To develop complexity theory, we need the algorithmic concept of nondeterminism.
- already known for Turing machines (⇝ chapter C7):
  - An NTM can have more than one possible successor configuration for a given configuration.
  - Input x is accepted if there is at least one possible computation (configuration sequence) that leads to an end state.
- Here we analogously introduce nondeterminism for pseudo-code.

German: Nichtdeterminismus

Nondeterministic Algorithms

nondeterministic algorithms:

- All constructs of deterministic algorithms are also allowed in nondeterministic algorithms: **IF**, **WHILE**, etc.
- Additionally, there is a nondeterministic assignment:
  GUESS $x_i \in \{0, 1\}$

  where x_i is a program variable.

German: nichtdeterministische Zuweisung

Nondeterministic Algorithms: Acceptance

- Meaning of **GUESS** $x_i \in \{0, 1\}$:
  x_i is assigned either the value 0 or the value 1.
- This implies that the behavior of the program on a given input is no longer uniquely defined: there are multiple possible execution paths.
- The program accepts a given input if at least one execution path leads to an **ACCEPT** statement.
- Otherwise, the input is rejected.

Note: asymmetry between accepting and rejecting!
(cf. semi-decidability)

More Complex GUESS Statements

- We will also guess more than one bit at a time:
  GUESS $x \in \{1, 2, \dots, n\}$

  or more generally
  GUESS $x \in S$

  for a set S.
- These are abbreviations and can be split into $\lceil \log_2 n \rceil$ (or $\lceil \log_2 |S| \rceil$) "atomic" **GUESS** statements.

Example: Nondeterministic Algorithms (1)

Example (DirHamiltonCycle)

input: directed graph $G = \langle V, E \rangle$

start := an arbitrary node from V
current := *start*
remaining := $V \setminus \{start\}$
WHILE $remaining \neq \emptyset$:
  GUESS $next \in remaining$
  IF $\langle current, next \rangle \notin E$:
    REJECT
  $remaining := remaining \setminus \{next\}$
  current := *next*
IF $\langle current, start \rangle \in E$:
  ACCEPT
ELSE:
  REJECT

Example: Nondeterministic Algorithms (2)

- With appropriate data structures, this algorithm solves the problem in $O(n \log n)$ program steps, where $n = |V| + |E|$ is the size of the input.
- How many steps would a deterministic algorithm need?

Guess and Check

- The DirHamiltonCycle example illustrates a general design principle for nondeterministic algorithms:

  guess and check
- In general, nondeterministic algorithms can solve a problem by first guessing a "solution" and then verifying that it is indeed a solution. (In the example, these two steps are interleaved.)
- If solutions to a problem can be efficiently verified, then the problem can also be efficiently solved if nondeterminism may be used.

German: Raten und Prüfen

The Power of Nondeterminism

- Nondeterministic algorithms are very powerful because they can "guess" the "correct" computation step.
- Or, interpreted differently: they go through many possible computations "in parallel", and it suffices if one of them is successful.
- Can they solve problems efficiently (in polynomial time) which deterministic algorithms cannot solve efficiently?
- This is the big question!

Summary (1)

- Complexity theory deals with the question which problems can be solved efficiently and which ones cannot.
- here: focus on what can be computed in polynomial time
- To formalize this, we use Turing machines, but other formalisms are polynomially equivalent.
- We consider decision problems, but the results directly transfer to general computational problems.

Summary (2)

important concept: nondeterminism

- Nondeterministic algorithms can "guess", i. e., perform multiple computations "at the same time".
- An input receives a "yes" answer if at least one computation path accepts it.
- in NTMs: with nondeterministic transitions ($\delta(q, a)$ contains multiple elements)
- in pseudo-code: with **GUESS** statements